AN EXPLICIT p-ADIC THEORY OF $\pi_1^{\mathrm{un,DR}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$

I : The Frobenius structure of $\pi_1^{\mathrm{un,DR}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$

I-2 : Indirect resolution of the equation of horizontality of Frobenius

DAVID JAROSSAY

Abstract. Let $X = \mathbb{P}^1 - (\{0, \infty\} \cup \mu_N) \,/\, W(\mathbb{F}_q)$, with $N \in \mathbb{N}^*$ and $\mathbb{F}_q$ of characteristic p prime to N and containing a primitive N-th root of unity. We establish an explicit theory of the crystalline Frobenius of the pro-unipotent fundamental groupoid of X.
In I, we compute explicitly the Frobenius structure and in particular the periods associated with it, i.e. cyclotomic p-adic multiple zeta values.
In this I-2, we provide an "indirect" resolution of the equation of horizontality of Frobenius, as two different formulas for the operation of multiplying by p the upper bound of weighted multiple harmonic sums. Both are expressed via a new operation of the pro-unipotent fundamental groupoid that we call the harmonic Ihara action. We build a torsor for the harmonic Ihara action containing the sequence of weighted multiple harmonic sums, and the identification between the two computations gives an indirect way to compute cyclotomic p-adic multiple zeta values and overconvergent hyperlogarithms.
This also gives an expression of prime weighted multiple harmonic sums in terms of cyclotomic p-adic multiple zeta values, and enlightens the nature of Kaneko-Zagier's finite multiple zeta values.

Contents

1. Introduction

Let p be a prime number, $N \in \mathbb{N}^{*}$ an integer prime to p, and $R = W(\mathbb{F}_q)$, the ring of Witt vectors of a finite field of characteristic p which contains a primitive N-th root of unity. We denote such a root by ξ, we denote also by $z_i = \xi^i$ for $i \in \{1, \ldots, N\}$, and by $K = \text{Frac}(R)$. Let X be the curve $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N) \ / \ R$, and X_K be the base change of X to K. By Deligne, [D] §10, the De Rham pro-unipotent fundamental groupoid $\pi_1^{\text{un,DR}}(X_K)$ is defined, and by [D], §11, it has a Frobenius structure. The variety X is also subject to Coleman integration in the sense of Besser [Bes] and Vologodsky [V].
In this part I, we want to compute the Frobenius structure of $\pi_1^{\text{un,DR}}(X_K)$, and in particular the periods associated with it ; they are iterated integrals, called p-adic cyclotomic multiple zeta values. For a more detailed introduction about the nature of these objects, see the introduction of I-1.

Let us recall the technical situation. The bundle of paths of $\pi_1^{\text{un,DR}}(X_K)$ starting at a given base-point, that we choose equal to $\vec{1}_0$ following the usual convention, is equipped with a canonical connexion denoted by ∇_{KZ}. Let f be the Frobenius of R ; since the z_i's are N-th roots of unity, we have $f(z_i) = z_i^p$, $i \in \{1, \ldots, N\}$. Let $X^{(p)}$ be the pull-back of X by f. Let X_K, resp. $X_K^{(p)}$ be the base change of X resp. $X^{(p)}$ to $\text{Spec}(K)$. There is a pull-back by Frobenius of the De Rham fundamental groupoid of $X_K^{(p)}$, constructed analytically, and denoted by $F^{*}\pi_1^{\text{un,DR}}(X_K^{(p)})$, such that there exists a unique morphism

$$F_{*} : \pi_1^{\text{un,DR}}(X_K) \xrightarrow{\sim} F^{*}\pi_1^{\text{un,DR}}(X_K^{(p)}) \tag{1.1}$$

that is horizontal with respect to the canonical connexions on the bundles of paths starting at a given base-point on both sides. It is an isomorphism. The crystalline Frobenius ϕ is the inverse of F_{*}. We have fixed for the whole of I-1 and I-2 an element $\alpha \in \mathbb{N}^{*}$ that serves as the power of Frobenius, i.e. we consider either F_{*} or $(p^{\alpha})^{\text{weight}}\phi^{\alpha}$.
Computing the action of Frobenius is equivalent to computing its restriction to canonical De Rham paths ; it is expressed by the couple $(\text{Li}_{p,\alpha}^{\dagger}, \zeta_{p,\alpha})$, where $\text{Li}_{p,\alpha}^{\dagger}$, called p-adic hyperlogarithm, is an overconvergent function on a certain rigid analytic subspace of $\mathbb{P}^{1,\text{an}}/K$, and $\zeta_{p,\alpha}$ denotes the cyclotomic p-adic multiple zeta values. This time, the convenient way to view the differential equation of horizontality of Frobenius is as an expression of $\text{Li}_{p,\alpha}^{\dagger}$ in terms of the other kind of p-adic hyperlogarithms Li_p^{KZ}, that are solutions to ∇_{KZ} :

$$\begin{aligned} \text{Li}_{p,\alpha}^{\dagger}(z)(e_0, e_{z_1}, \ldots, e_{z_N}) \times \text{Li}_{p,X_K^{(p^{\alpha})}}^{\text{KZ}}(z^{p^{\alpha}})\big(e_0, {\Phi_{p,\alpha}^{(z_1)}}^{-1} e_{z_1} \Phi_{p,\alpha}^{(z_1)}, \ldots, {\Phi_{p,\alpha}^{(z_N)}}^{-1} e_{z_N} \Phi_{p,\alpha}^{(z_N)}\big) \\ = \text{Li}_{p,X_K}^{\text{KZ}}(z)(p^{\alpha}e_0, p^{\alpha}e_{z_1}, \ldots, p^{\alpha}e_{z_N}) \end{aligned} \tag{1.2}$$

This equality holds in an algebra of Coleman functions. The results of explicit computation are all expressed through multiple harmonic sums, that are essentially the coefficients of the series expansion of Li_p^{KZ} at 0 : Let, for $n \in \mathbb{N}^{*}$, $d \in \mathbb{N}^{*}$, $(s_1, \ldots, s_d) \in (\mathbb{N}^{*})^d$, and $i_1, \ldots, i_{d+1} \in \{1, \ldots, N\}$; weighted mutiple harmonic sums are :

$$\text{har}_n \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_1 \\ s_d, \ldots, s_1 \end{array} \right) = n^{s_d + \ldots + s_1} \sum_{0 < n_1 < \ldots < n_d < n} \frac{\left(\frac{z_{i_2}}{z_{i_1}}\right)^{n_1} \ldots \left(\frac{z_{i_{d+1}}}{z_{i_d}}\right)^{n_d} \left(\frac{1}{z_{i_{d+1}}}\right)^{n}}{n_1^{s_1} \ldots n_d^{s_d}} \in K \tag{1.3}$$

We call prime multiple harmonic sums those whose upper bound n is a power of p.

In I-1, we have given an inductive computation of the couple $(\text{Li}_{p,\alpha}^{\dagger}, \zeta_{p,\alpha})$, that we have qualified

as "direct" ; in this I-2, we will give a computation of them of a different spirit, which we will qualify as "indirect". The methods are the following. In I-1, by induction on the weight and the depth, we computed the whole of the maps $\mathrm{Li}^{\dagger}_{p,\alpha}$ and we evaluated them at tangential base-points. In I-2, we will not employ at all the operation of evaluation at tangential base-points but, instead, the translation of equation (1.2) on the coefficients of the series expansion at the origin. There, $\mathrm{Li}^{\dagger}_{p,\alpha}$ and $\zeta_{p,\alpha}$ will appear as certain "coefficients" of certain transformations of multiple harmonic sums, variants of the Frobenius. In order to deduce from it an explicit formula for $\mathrm{Li}^{\dagger}_{p,\alpha}$ and $\zeta_{p,\alpha}$, we will write another version of the same transformations in a way that involves exclusively mutliple harmonic sums, thus for which the coefficients are explicit, and say that the coefficients of the two versions are equal.

Behind this method, there is a concept that we will define a future work and that we call the rational realization of the motivic pro-unipotent fundamental groupoid. We will say that the first type of transformations of multiple harmonic sums evoked above, whose coefficients will involve $\mathrm{Li}^{\dagger}_{p,\alpha}$ or $\zeta_{p,\alpha}$, lies in the "De Rham-rational" setting ; whereas the second type of transformations evoked above, that involves exclusively multiple harmonic sums, will be said to lie in the "rational" setting. We will think of the explicit formula for $(\mathrm{Li}^{\dagger}_{p,\alpha},\zeta_{p,\alpha})$, given by the identification between the De Rham-rational and rational computations, as arising from an "isomorphism of comparison" between the De Rham and rational realizations.

We recall that the Ihara action, or Ihara product, is a certain emanation of the motivic Galois action on the De Rham fundamental groupoid of X_K, in particular at the couples of tangential base-points $(\vec{1}_{z_i},\vec{1}_0)$, $i\in\{1,\ldots,N\}$, but also, in a larger sense, at all base-points. We use the terminology Goncharov coaction, or Goncharov coproduct, for the dual of the Ihara action, that is essentially equivalent to an operation computed by Goncharov in the Hodge realization. The Frobenius action is closely connected to the Ihara action of the particular element $(\mathrm{Li}^{\dagger}_{p,\alpha},\zeta_{p,\alpha})$.

The method relies on the following technical lemma : in a certain limit regarding the sequences of differential forms under iterated integration, the overconvergent factor $\mathrm{Li}^{\dagger}_{p,\alpha}$ of (1.2) becomes trivial (Lemma 3.2.2), whereas the limits of the other factors are non-trivial. By this fact, it will be possible to compute cyclotomic p-adic multiple zeta values $\zeta_{p,\alpha}$ without having to compute $\mathrm{Li}^{\dagger}_{p,\alpha}$ at the same time. Partly because of this simplification, the I-2 will provide more readable formulas for $\zeta_{p,\alpha}$, and also $\mathrm{Li}^{\dagger}_{p,\alpha}$, than I-1. Indeed, instead of having to solve a complicated system of equations, formed by (1.2) and a certain algebraic relation satisfied by the Frobenius, as we did in I-1, here we only have to write down two variants of the Frobenius action and compare them to each other. Note that the information on the limit of $\mathrm{Li}^{\dagger}_{p,\alpha}$ implying Lemma 3.2.2 relies entirely on the bounds of valuations of $\mathrm{Li}^{\dagger}_{p,\alpha}$ proved in Appendix to Theorem I-1, that proves nevertheless the utility of the method of I-1.

We will proceed in two steps, and compute first cyclotomic p-adic multiple zeta values (Theorem I-2.a), and then the overconvergent p-adic hyperlogarithms (Theorem I-2.b).

Below, $\tilde{w}/X_K$, resp. $\tilde{w}^{(p^\alpha)}/X_K^{(p^\alpha)}$ refers to the set of the indices $\left(\begin{array}{c} z_{i_{d+1}}^{p^\alpha},\ldots,z_1^{p^\alpha} \\ s_d,\ldots,s_1 \end{array}\right)$, resp. $\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_1 \\ s_d,\ldots,s_1 \end{array}\right)$ and $\Phi_{p,\alpha}=(p^\alpha)^{\text{weight}}\phi^\alpha({}_{\vec{1}_{z_i^{p^\alpha}}}1_{\vec{1}_0})$, $i\in\{1,\ldots,N\}$ is the generating series

of cyclotomic p-adic multiple zeta values. We denote by $\mathrm{har}_{p^{\alpha}\mathbb{N}} = \big((\mathrm{har}_{p^{\alpha}n}(\tilde{w}))_{\tilde{w}/X_K}\big)_{n\in\mathbb{N}}$, $\mathrm{har}_{\mathbb{N}}^{(p^{\alpha})} = \big((\mathrm{har}_{n}(\tilde{w}^{(p^{\alpha})}))_{\tilde{w}^{(p^{\alpha})}/X_K^{(p^{\alpha})}}\big)_{n\in\mathbb{N}}$, and $\mathrm{har}_{p^{\alpha}} = (\mathrm{har}_{p^{\alpha}}(w)_{w/X_K})$. We also denote by $\Pi_{1,0}(K) = \pi_1^{\mathrm{un,DR}}(X_K,\vec{1}_1,\vec{1}_0)(K)$.

For the first step, let us denote by $\circ^{\mathrm{DR}}$ the Ihara action at the tangential base-points $(\vec{1}_1,\vec{1}_0)$; $\Pi_{1,0}(K)$ is equipped with a topology that we will define in §2.4. We introduce here a new notion of "harmonic Ihara action".

Theorem I-2.a i) There exists a topological subgroup $\tilde{\tilde{\Pi}}_{1,0}(K)_{\Sigma} \subset \Pi_{1,0}(K)$ for $\circ^{\mathrm{DR}}$, and an explicit continuous group action, that we call the De Rham-rational harmonic Ihara action,

$$\circ_{\mathrm{har}}^{\mathrm{DR,RT}} : \tilde{\tilde{\Pi}}_{1,0}(K)_{\Sigma} \times \mathrm{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\mathrm{const}}) \to \mathrm{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$$

such that

$$\mathrm{har}_{p^{\alpha}\mathbb{N}} = \Phi_{p,\alpha} \ \circ_{\mathrm{har}}^{\mathrm{DR,RT}} \ \mathrm{har}_{\mathbb{N}}^{(p^{\alpha})}$$

ii) There exists a topological group $(\tilde{\tilde{\Pi}}_{1,0}^{\mathrm{RT}}(K)_{\Sigma},\circ^{\mathrm{RT}})$ and an explicit continuous group action, that we call the rational harmonic Ihara action,

$$\circ_{\mathrm{har}}^{\mathrm{RT,RT}} : \tilde{\tilde{\Pi}}_{1,0}^{\mathrm{RT}}(K)_{\Sigma} \times \mathrm{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\mathrm{const}}) \to \mathrm{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$$

and a map $\Sigma_{\mathrm{har}}^{\mathrm{RT}}$ such that

$$\mathrm{har}_{p^{\alpha}\mathbb{N}} = \Sigma_{\mathrm{har}}^{\mathrm{RT}}\,\mathrm{har}_{p^{\alpha}} \ \circ_{\mathrm{har}}^{\mathrm{RT,RT}} \ \mathrm{har}_{\mathbb{N}}^{(p^{\alpha})}$$

iii) There exists a torsor $\tilde{\mathcal{T}}_{\mathrm{har}}^{\mathrm{DR,RT}}$ for $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ and a torsor $\tilde{\mathcal{T}}_{\mathrm{har}}^{\mathrm{RT}}$ for $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$, both containing $\mathrm{har}_{\mathbb{N}}^{(p^{\alpha})}$, such that we have an isomorphism of torsors :

$$\mathrm{comp}_{\circ_{\mathrm{har}}} : \tilde{\mathcal{T}}_{\mathrm{har}}^{\mathrm{RT,RT}} \xrightarrow{\sim} \tilde{\mathcal{T}}_{\mathrm{har}}^{\mathrm{DR,RT}}$$

The definition of $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ is given in §3.3 and the definition of $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$ is given in §5.4. The reader used to the dual setting may want to consider instead the dual of the maps $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ and $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$ and call them the De Rham-rational resp. rational harmonic Goncharov coaction. The part iii) of the theorem implies that one can identify the p-adic coefficients of equalities of i) and ii) above, and that the identification yields an indirect computation of p-adic cyclotomic multiple zeta values, as

$$\Phi_{p,\alpha} = \Sigma_{\mathrm{har}}^{\mathrm{RT}}\,\mathrm{har}_{p^{\alpha}}$$

The harmonic Ihara action does not depend on p, and it depends on N only through certain rational functions applied to the primitive N-th root of unity ξ, as in I-1. This will be precised throughout the text. The justification of the terminology "harmonic Ihara action" will follow from the possibility of viewing it as a kind of motivic Galois action, that governs the link between the algebraic relations of p-adic multiple zeta values and the one of another kind of periods of $\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0,\mu_N,\infty\})$: this will be developed in part II.

Example a.1 : In depth one and for $\mathbb{P}^1 - \{0,1,\infty\}$, the i) and ii) of Theorem I-2.a imply, respectively, with $\mathcal{B}_b^l$ certain rational coefficients that we will define in §4 :

$$\mathrm{har}_{p^{\alpha}n}(s) = \mathrm{har}_n(s) + \sum_{b\in\mathbb{N}} n^{b+s}(\Phi_{p,\alpha}^{-1}e_1\Phi_{p,\alpha})[e_0^b e_1 e_0^{s-1}e_1]$$

$$\mathrm{har}_{p^\alpha n}(s) = \mathrm{har}_n(s) + \sum_{b\geq 1} n^{b+s} \sum_{l\geq b-1} \binom{-s}{l} \mathcal{B}_b^l \,\mathrm{har}_{p^\alpha}(s+l)$$

By identifying i) and ii) according to iii) on this example, we retrieve a formula given in I-1, §1. It is in depth > 1 that the combinatorics of I-1 and I-2 differ. In any depth, the equations for i) and ii) of Theorem I-2.a are indexed by the subsequences of the word $\tilde{w}$ in the argument and their associated quotient sequences, as is the formula for the dual of the usual Ihara action ; written that way, the equation for i) is extremely close to the usual easy formula for the dual of the Ihara action ; in addition, the map Σ^{RT}, i.e. the coefficients of ii) become explicit and readable after introducing (§4-§5) certain transformations of rational polynomials in several variables, that can be indexed by certain paths on a tree. Nevertheless, their utility is visible only in depths ≥ 3 ; in depth 2 these tools are not relevant since one can write the formula in a few lines, as we do below.

Example a.2 : In depth two and for $\mathbb{P}^1 - \{0,1,\infty\}$, without using the tools evoked above, the i) of Theorem I-2.a implies :

$$\begin{aligned}\mathrm{har}_{p^\alpha n}(s_2,s_1) = \mathrm{har}_n(s_2,s_1) &+ \textstyle\sum_{b\in\mathbb{N}} n^{b+s_2+s_1} (\Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha})[e_0^l e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] \\ &+ \textstyle\sum_{r_2=0}^{s_2-1} \mathrm{har}_n(s_2-r_2) n^{r_2+s_1} (\Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha})[e_0^{r_2} e_1 e_0^{s_1-1} e_1] \\ &+ \textstyle\sum_{r_1=0}^{s_1-1} \mathrm{har}_n(s_1-r_1) \sum_{b\in\mathbb{N}} n^{b+s_2+r_1} (\Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha})[e_0^b e_1 e_0^{s_2-1} e_1 e_0^{r_1}]\end{aligned}$$

The ii) of Theorem I-2.a implies, again with certain rational coefficients $\mathcal{B}$ that we will define in §4 :

$$\begin{aligned}\mathrm{har}_{p^\alpha n}(s_2,s_1) = \mathrm{har}_n(s_2,s_1) &+ \textstyle\sum_{t\geq 1} n^{s_1+s_2+t} \Bigg[\sum_{\substack{l_1,l_2\geq 0\\ l_1+l_2\geq t-2}} \mathcal{B}_t^{l_2,l_1} \prod_{i=1}^2 \binom{-s_i}{l_i} \mathrm{har}_{p^\alpha}(s_i+l_i) + \\ &\textstyle\sum_{\substack{l_1,l_2\geq 0\\ l_1+l_2\geq t-1}} \mathcal{B}_t^{l_1+l_2} \prod_{i=1}^2 \binom{-s_i}{l_i} \mathrm{har}_{p^\alpha}(s_2+l_2, s_1+l_1)\Bigg] + \\ &+ \textstyle\sum_{t\geq 1} n^{s_2+s_1+t} \sum_{l\geq t-1} \Bigg[\binom{-s_1}{l+s_2} \mathcal{B}_t^{l+s_2,-s_2} - \binom{-s_2}{l+s_1} \mathcal{B}_t^{l+s_1,-s_1} \Bigg] \\ &- n^{s_2+s_1} \Bigg[\textstyle\sum_{l_1\geq s_2-1} \mathcal{B}_{s_2}^{l_1} \binom{-s_1}{l_1} \mathrm{har}_{p^\alpha}(s_1+l_1) - \sum_{l_2\geq s_1-1} \mathcal{B}_{s_1}^{l_2} \binom{-s_2}{l_2} \mathrm{har}_{p^\alpha}(s_2+l_2) \Bigg] \\ &\textstyle\sum_{\substack{1\leq t<s_2\\ l\geq t-1}} n^{s_1+t} \mathrm{har}_n(s_2-t) \mathcal{B}_t^l \binom{-s_1}{l} \mathrm{har}_{p^\alpha}(s_1+l) - \sum_{\substack{1\leq t<s_1\\ l'\geq t-1}} n^{s_2+t} \mathrm{har}_n(s_1-t) \mathcal{B}_t^{l_2} \binom{-s_2}{l'} \mathrm{har}_{p^\alpha}(s_2+l')\end{aligned}$$

We obtain in particular, by iii) of the Theorem I-2.a, a formula for $(\Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha})[e_0^t e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1]$; it is different in the cases $t>0$ and $t=0$.

Shape of the formula for Σ^{RT}. One has a formula for Σ^{RT} (resp. for $\zeta_{p,\alpha}$) as a sum, where the summand is an explicit function of the argument (resp. of har_{p^α}), the coefficients $\binom{-s}{l}$ and $\mathcal{B}$, and the dommain of summation is also explicit, made out of the rational analogues of the operations of taking subwords of quotient words of words of e_Z, and of a set of paths on a tree, that arises from a map of "elimination of positive powers" of a certain generalization of multiple harmonic sums. This will be made precise by §4, §5 and §6.

The Theorem I-2.a implies the following concise expression of prime weighted multiple harmonic sums in terms of p-adic cyclotomic multiple zeta values :

Corollary I-2.a For all indices $(w,\tilde{w}) = \left(e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}}, \left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)\right)$, we

have :

$$(1.4)\qquad \mathrm{har}_{p^{\alpha}}(\tilde{w}) = (-1)^{d} \sum_{z \in Z - \{0,\infty\}} z^{-p^{\alpha}} \big({\Phi_{p,\alpha}^{(z)}}^{-1} e_{z} \Phi_{p,\alpha}^{(z)}\big)[\frac{1}{1-e_{0}}w]$$

$$= \sum_{\substack{d' \in \{1,\ldots,d+1\} \\ (l_{d'},\ldots,l_{d}) \in \mathbb{N}^{d-d'+1}}} z_{i_{d'}}^{-p^{\alpha}} \prod_{i=d'}^{d} \binom{-s_{i}}{l_{i}} (-1)^{s_{i}} \zeta_{p,\alpha}^{(z_{i_{d'}})} \left(\begin{array}{c} z_{i_{d'+1}} \;\ldots\; z_{i_{d+1}} \\ s_{d'}+l_{d'},\ldots,s_{d}+l_{d} \end{array} \right) \zeta_{p,\alpha}^{(z_{i_{d'}})} \left(\begin{array}{c} z_{i_{d'-1}},\ldots,z_{i_{1}} \\ s_{d'-1},\ldots,s_{1} \end{array} \right)$$

In particular, for $\mathbb{P}^{1} - \{0,1,\infty\}$, we have :

$$\mathrm{har}_{p^{\alpha}}(\tilde{w}) = (\Phi_{p,\alpha}^{-1} e_{1} \Phi_{p,\alpha})[\frac{1}{1-e_{0}}w]$$

$$= \sum_{d'=0}^{d} \sum_{l_{d'+1},\ldots,l_{d} \in \mathbb{N}} \prod_{i=d'}^{d} \binom{-s_{i}}{l_{i}} \zeta_{p,\alpha}(s_{d'+1}+l_{d'+1},\ldots,s_{d}+l_{d}) \zeta_{p,\alpha}(s_{d'},\ldots,s_{1})$$

We will give two ways of deducing this result from Theorem I-2.a, one from the De Rham-rational part i) and one from the rational part ii) and the De Rham-rational comparison iii). Both consist in considering the $n=1$ term of the Theorem I-2.a, and use that all multiple harmonic sums $\mathrm{har}_{n=1}$ (1.3) are empty. By this fact, when $n=1$, almost all terms of the formula of Theorem I-2.a i) vanish, and the Corollary I-2.a follows by the explicit formula for $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ described in §3.3. Secondly, if we denote by ${\Sigma_{\mathrm{har}}^{\mathrm{DR}}}^{-1}$ the map that transforms $\Phi_{p,\alpha}$ into the generating series of the numbers $\sum_{z \in Z-\{0,\infty\}} \big({\Phi_{p,\alpha}^{(z)}}^{-1} e_{z} \Phi_{p,\alpha}^{(z)}\big)[\frac{1}{1-e_{0}}w]$ on w, then, by considering the $n=1$ term in the rational computations of §4-§5 we can prove that

$${\Sigma_{\mathrm{har}}^{\mathrm{DR}}}^{-1} \circ \Sigma_{\mathrm{har}}^{\mathrm{RT}} = \mathrm{id}$$

and this gives Corollary I-2.a by Theorem I-2.a ii) and iii) : in more concrete terms, the Corollary I-2.a also follows by inverting the series expansion of $\zeta_{p,\alpha}$ in terms of $\mathrm{har}_{p^{\alpha}}$. Note that we will define in II-2 a De Rham harmonic Ihara action $\circ_{\mathrm{har}}^{\mathrm{DR,DR}}$, whose comparison with $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ will be governed by ${\Sigma_{\mathrm{har}}^{\mathrm{DR}}}^{-1}$.

The second step is now to compute $\mathrm{Li}_{p,\alpha}^{\dagger}$. The simplest solution is to say that, having computed $\zeta_{p,\alpha}$ by Theorem I-2.a and knowing that $\mathrm{Li}_{p}^{\mathrm{KZ}}$ is explicit on $]0[$, then we know an explicit formula for $\mathrm{Li}_{p,\alpha}^{\dagger}$ by the horizontality equation (1.2). However, we want to pursue our indirect method of the identification between a computation in the De Rham setting and an elementary computation involving multiple harmonic sums. Let us denote by U^{an} the affinoid rigid analytic space $(\mathbb{P}^{1,\mathrm{an}} - \cup_{i=1}^{N}]z_{i}[)/K$; the bundle of De Rham paths of $\pi_{1}^{\mathrm{un,DR}}(X_{K})(K)$ starting at $\vec{1}_{0}$ is a trivial torsor under $\Pi_{0,0}(K) = \pi_{1}^{\mathrm{un,DR}}(X_{K},\vec{1}_{0})(K)$, and then its group of rigid analytic sections on U^{an} is identified with $\Pi_{0,0}(\mathfrak{A}(U^{\mathrm{an}}))$ where $\mathfrak{A}(U^{\mathrm{an}})$ is the algebra of rigid analytic functions on U^{an}. The Ihara action on the whole De Rham groupoid provides in particular a map :

$$\circ_{U^{\mathrm{an}}}^{\mathrm{DR}} : \big(\Pi_{1,0}(K) \times \Pi_{0,0}(\mathfrak{A}(U^{\mathrm{an}}))\big) \times \Pi_{0,0}(\mathfrak{A}(U^{\mathrm{an}})) \rightarrow \Pi_{0,0}(\mathfrak{A}(U^{\mathrm{an}}))$$

By considering series expansions at 0 of elements of $\mathfrak{A}(U^{\mathrm{an}})$, i.e. by composing $\circ_{U^{\mathrm{an}}}^{\mathrm{DR}}$ with the isomorphism $\mathrm{comp}_{U^{\mathrm{an}}}$ of I-1 §4.1 and its inverse, we obtain its De Rham-rational analogue :

$$\circ_{U^{\mathrm{an}}}^{\mathrm{DR,RT}} : \big(\Pi_{1,0}(K) \times \Pi_{0,0}(\mathfrak{A}(U^{\mathrm{an}}))\big) \times \mathrm{Map}(\mathbb{N}, \mathfrak{A}(U^{\mathrm{an}})\langle\langle e_{Z}\rangle\rangle^{\mathrm{const}}) \rightarrow \mathrm{Map}(\mathbb{N}, \mathfrak{A}(U^{\mathrm{an}})\langle\langle e_{Z}\rangle\rangle^{\mathrm{const}})$$

On the other hand, the Frobenius action on $\Pi_{0,0}(\mathfrak{A}(U^{\mathrm{an}}))$ is essentially the Ihara action of $(\mathrm{Li}_{p,\alpha}^{\dagger}, \zeta_{p,\alpha})$ pre-composed with $f(z) \mapsto f(z^{p^{\alpha}})$, and the equation of horizontality of Frobenius

(1.2) reformulates as :

$$\tau(p^{\alpha})\,\mathrm{Li}^{\mathrm{KZ}}_{p,X_K}(z) = (\mathrm{Li}^{\dagger}_{p,\alpha},\zeta_{p,\alpha}) \circ^{\mathrm{DR}}_{U^{\mathrm{an}}} \mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^{\alpha})}}(z^{p^{\alpha}})$$

These facts imply directly, by definition, that :
i) We have

$$\mathrm{har}_{\mathbb{N}} = (\mathrm{Li}^{\dagger}_{p,\alpha},\zeta_{p,\alpha}) \circ^{\mathrm{DR,RT}}_{U^{\mathrm{an}}} \mathrm{har}^{(p^{\alpha})}_{\mathbb{N}}$$

The second main result of this paper is the following :

Theorem I-2.b ii) There exists an explicit continuous group action, that we call the rational Ihara action on bundle of paths starting at $\vec{1}_0$:

$$\circ^{\mathrm{RT,RT}}_{U^{\mathrm{an}}} : \tilde{\tilde{\Pi}}^{\mathrm{RT}}_{1,0}(K)_{\Sigma} \times \tilde{\tilde{\Pi}}^{\mathrm{RT}}_{0,0}(\mathfrak{A}(U^{\mathrm{an}}))_{\Sigma} \times \mathrm{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\mathrm{const}}) \rightarrow \mathrm{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$$

and a map $\Sigma^{\mathrm{RT}}_{U^{\mathrm{an}}}$ such that we have

$$\mathrm{har}_{\mathbb{N}} = \big(\Sigma^{\mathrm{RT}}_{U^{\mathrm{an}}}\,\mathrm{har}^{(p^{\alpha})}_{\mathbb{N}}, \Sigma^{\mathrm{RT}}\,\mathrm{har}_{p^{\alpha}}\big) \circ^{\mathrm{RT}}_{U^{\mathrm{an}}} \mathrm{har}^{(p^{\alpha})}_{\mathbb{N}}$$

iii) There exists a torsor $\tilde{\mathcal{T}}^{\mathrm{DR,RT}}_{\mathrm{har},U^{\mathrm{an}}}$ for $\circ^{\mathrm{DR,RT}}_{U^{\mathrm{an}}}$ and a torsor $\tilde{\mathcal{T}}^{\mathrm{RT,RT}}_{\mathrm{har},U^{\mathrm{an}}}$ for $\circ^{\mathrm{RT,RT}}_{U^{\mathrm{an}}}$, both containing $\mathrm{har}^{(p^{\alpha})}_{\mathbb{N}}$, such that we have an isomorphism of torsors :

$$\mathrm{comp}_{\circ_{\mathrm{har},U^{\mathrm{an}}}} : \tilde{\mathcal{T}}^{\mathrm{RT,RT}}_{\mathrm{har},U^{\mathrm{an}}} \xrightarrow{\sim} \tilde{\mathcal{T}}^{\mathrm{DR,RT}}_{\mathrm{har},U^{\mathrm{an}}}$$

The maps $\circ^{\mathrm{RT,RT}}_{U^{\mathrm{an}}}$ and $\Sigma^{\mathrm{RT}}_{U^{\mathrm{an}}}$ of ii) above are defined from the maps $\circ^{\mathrm{RT,RT}}_{\mathrm{har}}$ and $\Sigma^{\mathrm{RT}}_{\mathrm{har}}$ of Theorem I-2.a ii) by rewriting them as depending on an additional index that was hidden in their initial formulation, generalizing them from $\mathrm{har}_{p^{\alpha}\mathbb{N}}$ to $\mathrm{har}_{\mathbb{N}}$, and then by associating to them an infinite sequences of operators of "truncations".

The Theorem I-2.a and Theorem I-2.b combined form an indirect computation of the couple $(\mathrm{Li}^{\dagger}_{p,\alpha},\zeta_{p,\alpha})$, and thus an indirect computation of the Frobenius action on $\pi_1^{\mathrm{un,DR}}(\mathbb{P}^1 - \{0,\mu_N,\infty\})$.

The Corollary I-2.a in the case of $\mathbb{P}^1 - \{0,1,\infty\}$ and $\alpha = 1$ solves a conjecture of M. Hirose and S. Yasuda [Y]. This conjecture had been proved by M.Hirose in depth two. In depth one, the same conjecture is a restatement of the following result of Washington (particular case of Theorem 1 of [W]) : let L_p be the Kubota-Leopoldt zeta function, and let ω be the Teichmüller character ; then we have , for all $s \in \mathbb{N}^*$, $\mathrm{har}_p(s) = \sum_{l\geq 1}\binom{s}{-l} L_p(s+l,\omega^{1-s-l})$. Indeed, by a result of Coleman ([Co], I, equation (4)), we have, for all $s \in \mathbb{N}^*$, $\zeta_{p,1}(s) = p^s L_p(s,\omega^{1-s})$. We have announced the Theorem I-2.a i) and the principle of its proof in the case of $\mathbb{P}^1 - \{0,1,\infty\}$ in the note [J3].

Outline. The De Rham part of this paper is achieved in §2 and §3 ; the rational part in §4 and §5 ; the comparison between the two in §6. In §2, we establish a topological and algebraic setting to deal with certain aspects of the motivic Galois and Frobenius action on the pro-unipotent fundamental groupoid at tangential base-points. In §3, we prove the part i) of Theorem I-2.a, using the Appendix to Theorem I-1 as a key technical lemma (§3.2) and defining the De Rham-rational Ihara action (§3.3). In §4 and §5 we make very few references to the pro-unipotent fundamental groupoid. The setting is defined in §4 and the computations are made in §5, proving most of the rational part of Theorem I-2.a and Theorem I-2.b. The rational harmonic Ihara action is defined in §5.3. The rest of the proof of the Theorem I-2.a and Theorem I-2.b is made in §6.

In Appendix 1, we review an application of Corollary I-2.a to Kaneko-Zagier's conjecture on finite multiple zeta values and we make some related remarks.

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2. Properties of the motivic Galois action and Frobenius on $\pi_1^{\text{un,DR}}(X_K, \vec{1}_z, \vec{1}_0)$

In §2.1, we review some facts and notations concerning the De Rham fundamental groupoid of X_K at tangential base points, including the Ihara action, the action of $\mathbb{G}_m$ reflecting the weight, and the Frobenius, in particular the definition of cyclotomic p-adic multiple zeta values. In §2.2 we review some properties of subschemes of $\tilde{\Pi}_{z,0}$ of $\pi_1^{\text{un,DR}}(X_K, \vec{1}_z, \vec{1}_0) = \Pi_{z,0}$ defined by certain vanishing conditions in weight one. In §2.3 we describe, in terms the combinatorics of words on e_Z, the dual of usual operations on $K\langle\langle e_Z\rangle\rangle$. In §2.4, we consider $A\langle\langle e_Z\rangle\rangle$, $\Pi_{z,0}(A)$ for any complete K-normed algebra A, as well as certain of its vector subspaces, resp. subgroups, and we equip them with topologies ; we study the compatibilies between the usual algebraic operations and these topologies.

2.1. **Preliminaries.**

2.1.1. *The De Rham unipotent fundamental groupoid : notations.* The De Rham realization of the unipotent fundamental groupoid of X_K ([D], §12) will be denoted by $\pi_1^{\text{un,DR}}(X_K)$. It is a groupoid of affine schemes on X_K ; it is defined over $\mathbb{Q}$ although X_K is not. Aside from the points of X_K, the base-points of $\pi_1^{\text{un,DR}}(X_K)$ include also the tangentials base-points $\vec{v}_x$ (the tangent vector $\vec{v} \neq 0$ at a point $x \in Z$) and the canonical base-point ω_{DR}. Let e_Z be the alphabet $\{e_0, e_{z_1}, \ldots, e_{z_N}\}$. If $Z = \{0, 1, \infty\}$, this is $\{e_0, e_1\}$. Let also $e_\infty = -e_0 - \sum_{i=1}^N e_{z_i}$. The shuffle Hopf algebra over $\mathbb{Q}$ associated with e_Z is denoted by $\mathcal{O}^{ш, e_Z}$. It is freely generated, as a $\mathbb{Q}$-vector space, by the words on e_Z, including the empty word. Its product, the shuffle product, is denoted by ш. The weight of a word on e_Z is its number of letters. The depth, or total depth, of a word w on e_Z is its number of letters distinct from e_0. For $i \in \{1, \ldots, N\}$, the z_i-depth of a word w on e_Z is its number of letters equal to e_{z_i}. The pro-unipotent affine group scheme $\pi_1^{\text{un,DR}}(X_K, \omega_{\text{DR}})$ is equal to $\text{Spec}(\mathcal{O}^{ш, e_Z})$. The canonical De Rham path of each $\pi_1^{\text{un,DR}}(X_K, v, u)$ is denoted by ${}_v1_u$. Canonical paths are compatible to the groupoid structure. The scheme $\pi_1^{\text{un,DR}}(X_K, \omega_{\text{DR}})$ is canonically isomorphic to each $\pi_1^{\text{un,DR}}(X_K, x, y)$, and the isomorphisms are compatible with the groupoid structure ; the non-commutative algebra of formal power series with variables in e_Z, and coefficients in K is denoted by

$$K\langle\langle e_Z\rangle\rangle = K\langle\langle e_0, e_{z_1} \ldots, e_{z_N}\rangle\rangle$$

An element f of $K\langle\langle e_Z\rangle\rangle$ can be written uniquely as

$$f[\emptyset] + \sum_{\substack{d \in \mathbb{N}^* \\ (s_d, \ldots, s_0) \in (\mathbb{N}^*)^{d+1} \\ (i_d, \ldots, i_1) \in (Z - \{0, \infty\})^d}} f[e_0^{s_d - 1} e_{z_{i_d}} \ldots e_0^{s_1 - 1} e_{z_{i_1}} e_0^{s_0 - 1}] e_0^{s_d - 1} e_{z_{i_d}} \ldots e_0^{s_1 - 1} e_{z_{i_1}} e_0^{s_0 - 1}$$

For each algebra A, we have a (functorial) inclusion

$$\pi_1^{\text{un,DR}}(X_K, \omega_{\text{DR}})(A) \subset A\langle\langle e_Z\rangle\rangle$$

more precisely, the elements of $\pi_1^{\mathrm{un,DR}}(X_K,\omega_{\mathrm{DR}})(A)$ are the formal power series f satisfying $f[\emptyset]=1$ and the shuffle equation, that is to say $f[w ш w']=f[w]f[w']$ for all words $w,w'\in\mathcal{O}^{ш,e_Z}$.

We will consider the bundle of paths on the De Rham fundamental groupoid of X_K starting at $\vec{1}_0$, trivialized at $\vec{1}_0$. Its canonical connexion, named after Knizhnik and Zamolodchikov, can be written as :

$$\nabla_{\mathrm{KZ}} : f \mapsto f^{-1}\big(df - \tfrac{dz}{z}e_0 f + \textstyle\sum_{i=1}^{N} \tfrac{dz}{z-z_i} e_{z_i} f\big)$$

Let $\mathrm{Li}^{\mathrm{KZ}}_{p,X_K}$, resp. $\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}$ be the unique Coleman function on X_K, resp. $X_K^{(p^\alpha)}$, that is a horizontal section of the canonical connexion ∇_{KZ} associated with $\pi_1^{\mathrm{un,DR}}(X_K)$, resp. $\pi_1^{\mathrm{un,DR}}(X_K^{(p^\alpha)})$ (§1.1), with the asymptotics $\mathrm{Li}^{\mathrm{KZ}}_{p,X_K}(z) \sim_{z\rightarrow 0} e^{e_0 \log_p(z)}$, resp. $\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}(z) \sim_{z\rightarrow 0} e^{e_0 \log_p(z)}$.

2.1.2. *Motivic Galois action on* $\pi_1^{\mathrm{un,DR}}(X_K)$.

2.1.2.a. **The projective line minus a finite number of points over a number field**

Let a curve Y of the type $\mathbb{P}^1$ minus a finite number of points over a number field L ; the motivic fundamental groupoid of such a curve is defined by Deligne and Goncharov ([DG], §4). It is a groupoid on Y, including rational tangential base-points, in the category of affine schemes in the Tannakian category of mixed Tate motives over L, in the sense of [DG], §2. Let ω be the De Rham realization functor of the category of mixed Tate motives over L. The associated Tannakian group $G=\mathrm{Aut}^{\otimes}(\omega)$, that is called a motivic Galois group, and that acts on the De Rham unipotent fundamental groupoid of Y, admits a semi-direct product decomposition

$$G^{\omega} = \mathbb{G}_m \ltimes U^{\omega}$$

where U^{ω} is a pro-unipotent algebraic group. The action of $\mathbb{G}_m$ on $\pi_1^{\mathrm{un,DR}}(Y)$ is given by

$$\lambda \mapsto \tau(\lambda) = \text{ multiplication by } \lambda^{\text{weight}}$$

2.1.2.b. **Roots of unity**

Now assume that Y is $\mathbb{P}^1-\{0,\mu_N,\infty\}$ over $L=\mathbb{Q}[\mu_N]$, and denote $z_i=\xi^i$, $i\in\{1,\ldots,N\}$, where $\xi\in\mathbb{Q}[\mu_N]$ is a primitive N-th root of unity. We will use frequently the following notation originated in [DG], §5 :

Notation 2.1.1. For $x,y\in\{0\}\cup\mu_N(\overline{\mathbb{Q}})$, let $\Pi_{y,x}$ be the scheme $\pi_1^{\mathrm{un,DR}}(Y,\vec{1}_y,\vec{1}_x)$.

Notation 2.1.2. Let $\mathrm{Diag}^{\xi}\left(\prod_{i=1}^{N}\Pi_{z_i,0}\right)$ be the subscheme of $\prod_{i=1}^{N}\Pi_{z_i,0}$ made of the points $(g_{z_1},\ldots,g_{z_N})$ such that, for all $i\in\{1,\ldots,N\}$, we have : $g_{z_i}=(x\mapsto z_i x)_* g_{z_N}$.

We have canonical isomorphisms $\mathrm{Diag}^{\xi}\left(\prod_{i=1}^{N}\Pi_{z_i,0}\right)\simeq\Pi_{z_i,0}$. The action of U^{ω} on each $\Pi_{z,0}$ factorizes through an operation

$$\circ^{\mathrm{DR}} : \Pi_{z,0}\times\Pi_{z,0}\rightarrow\Pi_{z,0}$$

that is called the Ihara action, and is explicitly the following map :

$$g_z \circ^{\mathrm{DR}} f_z = g_z.f_z(e_0, g_{z_1}^{-1}e_{z_1}g_{z_1},\ldots,g_{z_N}^{-1}e_{z_N}g_{z_N}) \tag{2.1.1}$$

where $(g_{z_1},\ldots,g_{z_N})$ is the preimage in $\mathrm{Diag}^{\xi}\left(\prod_{i=1}^{N}\Pi_{z_i,0}\right)$ of g_z. Moreover, in this case of roots of unity, $\circ^{\mathrm{DR}}$ defines a group law on each $\Pi_{z,0}$ called the Ihara group law ([DG], §5).

2.1.2.c. **At all base-points**

Let z any point of X_K, and $\Pi_{z,0} = \pi_1^{\text{un,DR}}(X_K, z, \vec{1}_0)$. The Ihara action in a generalized sense is the following map :

$$(\Pi_{1,0} \times \Pi_{z,0}) \times (\Pi_{1,0} \times \Pi_{z,0}) \to (\Pi_{1,0} \times \Pi_{z,0})$$
$$(g_1, g_z) \times (f_1, f_z) \mapsto (g_1 \circ^{\text{DR}} f_1, g_z.f_z(e_0, g_{z_1}^{-1} e_{z_1} g_{z_1}, \ldots, g_{z_N}^{-1} e_{z_N} g_{z_N}))$$

where $g_{z_i} = (x \mapsto z_i x)_*(g)$ for $i \in \{1, \ldots, N\}$. For our purposes, we will consider z varying among points of the rigid analytic affinoid space $U^{\text{an}} = (\mathbb{P}^{1,\text{an}} - \cup_{i=1}^N]z_i[)/K$, and we will obtain a map as above where $\Pi_{z,0}$ is replaced by the group of rigid analytic sections on U^{an} of the bundle of paths of $\pi_1^{\text{un,DR}}$ starting at $\vec{1}_0$. For convenience of the notations we will forget the variable f_1, denote this map by $\circ_{U^{\text{an}}}^{\text{DR}}$, and we will denote $\circ_{U^{\text{an}}}^{\text{DR}}\big((g_1, g)\ ,\ f\big)$ by $(g_1, g) \circ_{U^{\text{an}}}^{\text{DR}} f$.

2.1.3. *The Frobenius structure of* $\pi_1^{\text{un,DR}}(X_K)$. Let $\alpha \in \mathbb{N}^*$, and let $F_*^\alpha : \pi_1^{\text{un,DR}}(X_K) \xrightarrow{\sim} (F^\alpha)^* \pi_1^{\text{un,DR}}(X_K^{(p^\alpha)})$ be the α-th power of the Frobenius map constructed [D], §11. The crystalline Frobenius of $\pi_1^{\text{un,DR}}(X_K)$ is defined as $\phi = F_*^{-1}$ [D], §13. We will consider either F_*^α or $\tau(p^\alpha)\phi^\alpha$.
For each $z \in Z - \{0\}$, let :

$$\begin{cases} \Phi_{p,-\alpha}^{(z^{p^\alpha})} = F_*^\alpha({}_{\vec{1}_z}1_{\vec{1}_0}) \in \pi_1^{\text{un,DR}}(X_K^{(p^\alpha)}, \vec{1}_{z^{p^\alpha}}, \vec{1}_0)(K) \\ \Phi_{p,\alpha}^{(z)} = \tau(p^\alpha)\phi^\alpha({}_{\vec{1}_{z^{p^\alpha}}}1_{\vec{1}_0}) \in \pi_1^{\text{un,DR}}(X_K, \vec{1}_z, \vec{1}_0)(K) \end{cases}$$

We also denote by $\Phi_{p,\alpha} = \Phi_{p,\alpha}^{(1)}$, $\Phi_{p,-\alpha} = \Phi_{p,-\alpha}^{(1)}$ $\Phi_p^{(z)} = \Phi_{p,1}^{(z)}$ and $\Phi_p = \Phi_{p,1}^{(1)}$.
The p-adic cyclotomic multiple zeta values are the following numbers : for $d \in \mathbb{N}^*$, $s_d, \ldots, s_1 \in \mathbb{N}^*$, and $i_{d+1}, \ldots, i_1 \in \{1, \ldots, N\}$, and $\alpha \in \mathbb{N}^*$,

$$\zeta_{p,\alpha}^{(z_{i_{d+1}})}\Big(\begin{array}{c} z_{i_d}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \Big) = (-1)^d \Phi_{p,\alpha}^{(z_{i_{d+1}})}[e_0^{s_d - 1} e_{z_{i_d}} \ldots e_0^{s_1 - 1} e_{z_{i_1}}]$$

$$\zeta_{p,\alpha}^{(z_{i_{d+1}}^{p^\alpha})}\Big(\begin{array}{c} z_{i_d}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \Big) = (-1)^d \Phi_{p,-\alpha}^{(z_{i_{d+1}}^{p^\alpha})}[e_0^{s_d - 1} e_{z_{i_d}} \ldots e_0^{s_1 - 1} e_{z_{i_1}}]$$

We also denote, by $\zeta_{p,\alpha} = \zeta(1)_{p,\alpha}$, $\zeta_{p,-\alpha} = \zeta(1)_{p,-\alpha}$, $\zeta(z)_p = \zeta(z)_{p,1}$ and $\zeta_p = \zeta_{p,1}$.
When $Z = \{0, 1, \infty\}$, we obtain the p-adic multiple zeta values :

$$\zeta_{p,\alpha}(s_d, \ldots, s_1) = (-1)^d \Phi_{p,\alpha}[e_0^{s_d - 1} e_1 \ldots e_0^{s_1 - 1} e_1]$$

One defines a generating series of overconvergent functions on $U^{\text{an}} = \mathbb{P}^{1,\text{an}} - \cup_{i=1}^N]z_i[$ over K by :

$$\mathrm{Li}_{p,\alpha}^\dagger(z) = \tau(p^\alpha)\phi^\alpha({}_z 1_{\vec{1}_0}) \in \Pi_{0,0}(\mathfrak{A}^\dagger(U^{\text{an}}))$$

with ${}_z 1_{\vec{1}_0}$ replaced by ${}_{\vec{1}_z} 1_{\vec{1}_0}$ if $z = \infty$.
For the purposes of the present paper, the most convenient formulation of the equation of horizontality of F_* is the following :

$$\text{(2.1.2)} \quad \mathrm{Li}_{p,\alpha}^\dagger(z)(e_0, e_{z_1}, \ldots, e_{z_N}) \times \mathrm{Li}_{p,X_K^{(p^\alpha)}}^{\text{KZ}}(z^{p^\alpha})\big(e_0, \Phi_{p,\alpha}^{(z_1)}{}^{-1} e_{z_1} \Phi_{p,\alpha}^{(z_1)}, \ldots, \Phi_{p,\alpha}^{(z_N)}{}^{-1} e_{z_N} \Phi_{p,\alpha}^{(z_N)}\big)$$
$$= \mathrm{Li}_{p,X_K}^{\text{KZ}}(z)(p^\alpha e_0, p^\alpha e_{z_1}, \ldots, p^\alpha e_{z_N})$$

Note that, if we choose to work with the inverse $\phi^{-\alpha}$ of the Frobenius, it would give the same result (Theorem I-2.a i) and Theorem I-2.b i)) but as an expression of har_n in terms of $\mathrm{har}_{p^\alpha n}$.

2.2. **A subscheme of $\Pi_{z,0}$ defined by vanishing conditions in weight one.** Let $z \in Z - \{0, \infty\}$.

Definition 2.2.1. Let $\tilde{\Pi}_{z,0}$ be the subscheme of $\Pi_{z,0}$ whose points are the grouplike series on e_Z satisfying :

$$f[e_0] = f[e_z] = 0$$

Lemma 2.2.2. i) $\tilde{\Pi}_{z,0}$ is a sub-group scheme of $\Pi_{z,0}$ for the usual group scheme structure on $\Pi_{z,0}$.
ii) The isomorphisms $\mathrm{Diag}^{\xi}\left(\prod_{i=1}^{N}\Pi_{z_i,0}\right) \simeq \Pi_{z,0}$ restrict to isomorphisms
$\mathrm{Diag}^{\xi}\left(\prod_{i=1}^{N}\tilde{\Pi}_{z_i,0}\right) \simeq \tilde{\Pi}_{z,0}$ and each $\tilde{\Pi}_{z,0}$ is a sub-group scheme of $\Pi_{z,0}$ for the Ihara product.

Proof. i) Direct consequence of the formulas for those two operations.
ii) Each $\tilde{\Pi}_{z,0}$ is the image by $(x \mapsto zx)_*$ of $\tilde{\Pi}_{1,0}$ and the rest follows from the formula (2.1.1) for the Ihara product. ☐

We saw in part I-1 that we have $\Phi^{(z)}_{p,\alpha} \in \tilde{\Pi}_{z,0}(K)$ for all $\alpha \in \mathbb{N}^*$. A reason for considering intrinsically the schemes $\tilde{\Pi}_{z,0}$ is their following property regarding the adjoint action $\mathrm{Ad}(e_z)$: $\Pi_{z,0} \to \mathrm{Lie}^{\vee}\,\Pi_{0,0}$ - note that in our convention of reading the multiplication of the groupoid $\pi_1^{\mathrm{un,DR}}(X_K)$ from the right to the left, $\mathrm{Ad}(e_z)$ is given on the points by $f \mapsto f^{-1}e_z f$.

Proposition 2.2.3. $\mathrm{Ad}(e_z)$ induces an injective map $\tilde{\Pi}_{z,0}(K) \hookrightarrow \mathrm{Lie}^{\vee}\,\Pi_{0,0}(K)$.

2.3. **Duals of the composition of power series on e_Z, adjoint action and Ihara action on $\Pi_{z,0}$.**

2.3.1. *Composition of series.* Let $f, g_{z_1}, \ldots, g_{z_N} \in K\langle\langle e_Z\rangle\rangle$. We want to describe the coefficients of the composition of formal series.

$$f(e_0, g_{z_1}, \ldots, g_{z_N})$$

We assume that $g_{z_1}[\emptyset] = \ldots = g_{z_N}[\emptyset] = 0$, and $f[e_0^m] = g_{z_1}[e_0^m] = \ldots = g_{z_N}[e_0^m] = 0$ for all $m \in \mathbb{N}^*$. This assumption will always be verified in practice ; it is not necessary to assume that f and the g_{z_i}'s are grouplike or primitive.

Let us fix a word w on e_Z. The coefficient $f(e_0, g_{z_1}, \ldots, g_{z_N})[w]$ will be expressed in terms of coefficients $g_{z_i}[w']$ with w' subwords of w, and $f[w'']$ with w'' certain "quotient words" of w, in the sense below.

Definition 2.3.1. Let $\mathcal{SW}(w)$ be the set of subwords of w that contain all the letters of w that are not e_0.
Let $sw \in \mathcal{SW}(w)$. A "connected partition" $(sw_j)_{j\in J}$ of sw is a partition of sw (viewed as the set of its letters), into subwords as $sw = \amalg_{j\in J} sw_j$, such that the letters of each sw_j are consecutive in sw (we will say that each sw_j is "connected" in sw), and such that at least one letter of each sw_j is not equal to e_0.
We say that a subword $sw \in \mathcal{SW}(w)$ is maximally at the left of w if it contains the letter of w distinct from e_0 that is the furthest to the right in w.

Definition 2.3.2. A coloring of a connected partition $(sw_j)_{j\in J}$ of an element sw of $\mathcal{SW}(w)$ is a map $J \mapsto \{z_1, \ldots, z_N\}$, that we will denote by $j \mapsto z_{i(j)}$.

Definition 2.3.3. Let $sw \in \mathcal{SW}(w)$, $(sw_j)_{j\in J}$ a connected partition of sw and $C = (z_{i(j)})_{j\in J}$ be a coloring of $(sw_j)_{j\in J}$. We call the quotient of w by the partitioned subword $sw = \amalg_{j\in J} sw_j$ colored in C and denote by $\frac{w}{((sw_j)_{j\in J}, C)}$ the word obtained by replacing, in w, each subword sw_j by the letter $e_{z_{i(j)}}$.

Lemma 2.3.4. We have, for all $w \in \mathcal{W}(e_Z)$, $sw \in \mathcal{SW}(w)$, $(sw_j)_{j\in J}$ connected partition of sw, and C coloring of $(sw_j)_{j\in J}$:

$$\mathrm{depth}(\tfrac{w}{((sw_j)_{j\in J},C)}) = \mathrm{depth}(w) - \textstyle\sum_{j\in J}(\mathrm{depth}(w_j)-1)$$

Proof. Clear. □

Proposition 2.3.5. We have :

$$f(e_0,g_{z_1},\ldots,g_{z_N})[w] = f[\emptyset] + \sum_{sw\in\mathcal{SW}(w)} \sum_{\substack{(sw_j)_{j\in J} \\ \text{connected} \\ \text{partition} \\ \text{of } sw}} \sum_{\substack{C=(z_{i(j)})_{j\in J} \\ \text{coloring of} \\ (sw_j)_{j\in J}}} \Big(\prod_{j\in J} g_{z_{i(j)}}[sw_j]\Big) f\Big[\frac{w}{((sw_j)_{j\in J},C)}\Big]$$

Proof. We write

$$f(e_0,g_{z_1},\ldots,g_{z_N}) = f[\emptyset] + \sum_{\substack{d\in\mathbb{N}^* \\ (s_d,\ldots,s_0)\in(\mathbb{N}^*)^{d+1} \\ (z_{i_d},\ldots,z_{i_1})\in(Z-\{0,\infty\})^d}} \Bigg($$

$$f[e_0^{s_d-1}e_{z_{i_d}}\ldots e_{z_{i_1}}e_0^{s_0-1}]e_0^{s_d-1}\big(\textstyle\sum_{w_d\in\mathcal{W}(e_Z)} g_{z_{i_d}}[w_d]w_d\big)\ldots\big(\textstyle\sum_{w_1\in\mathcal{W}(e_Z)} g_{z_{i_1}}[w_1]w_1\big)e_0^{s_0-1}\Bigg)$$

and we reindex the right-hand side by the words on e_Z. □

In the case of $\mathbb{P}^1-\{0,1,\infty\}$, the notion of coloring (Definition 2.3.2) becomes trivial ; if w is a word on $\{e_0,e_1\}$, $\mathcal{SW}(w)$ is the set of subwords of w that contain all its letters e_1. The formula of Proposition 2.3.5 is simplified into :

$$f(e_0,g)[w] = \sum_{sw\in\mathcal{SW}(w)} \sum_{\substack{(sw_j)_{j\in J}\text{ connected} \\ \text{partition of } sw}} \Big(\prod_{j\in J} g[sw_j]\Big) f\Big[\frac{w}{(sw_j)_{j\in J}}\Big]$$

2.3.2. *Adjoint action.* It is not necessary for our purposes to write a formula for the dual of the adjoint action, but we recall from I-1, §4 that it transforms the weight and the depth as follows :

Proposition 2.3.6. For each $n,d\in\mathbb{N}^*$, the dual of the map $f\in\tilde{\Pi}_{z,0}(K)\mapsto \mathrm{Ad}_f(e_Z)\in \mathrm{Lie}^\vee\,\Pi_{0,0}(K)$ restricts to a map

$$\mathrm{Ad}(e_Z)^\vee : \mathcal{W}_{n,d}(e_Z)\to\mathbb{Z}[\mathcal{W}_{n-1,d-1}(e_Z)]$$

2.3.3. *Ihara product.* As for the adjoint action, we do not need to write explicitly the formula for the dual, but we will use the following :

Corollary 2.3.7. The dual $(\circ^{\mathrm{DR}})^\vee : \mathcal{O}^{ш,e_Z}\to\mathcal{O}^{ш,e_Z}\otimes\mathcal{O}^{ш,e_Z}$ of the Ihara product on $\Pi_{z,0}(K)$ is compatible with the (weight grading, the) depth filtration and the integrality of coefficients, in the sense that, for all $n,d\in\mathbb{N}^*$, it restricts to a map :

$$(\circ^{\mathrm{DR}})^\vee : \mathcal{W}_{n,d}(e_Z)\to\oplus_{\substack{n_1+n_2=n \\ d_1+d_2=d}}\mathbb{Z}[\mathcal{W}_{n_1,d_1}(e_Z)]\otimes\mathbb{Z}[\mathcal{W}_{n_1,d_1}(e_Z)]$$

Proof. Direct combination of the equation (2.1.1) for the Ihara product, of the description of the dual of the composition of series (Proposition 2.3.5), the information on the depth of quotient words (Lemma 2.3.4), and on the information on the adjoint action (Proposition 2.3.6). □

2.4. **Topologies on $A\langle\langle e_Z\rangle\rangle$ and $\Pi_{z,0}(A)$.** We fix A any (ultrametric) complete normed K-algebra. The following facts will permit us to express in a compact way the bounds on the valuation of overconvergent p-adic hyperlogarithms and p-adic multiple zeta values of part I (§3.2). They are also a natural framework for all the infinite summations of p-adic multiple zeta values and multiple harmonic sums appearing in this work, and will allow us to define the De Rham-rational harmonic Ihara action as an topological action of a complete topological group (§3.3). Actually, their main applications will be in I-3.

2.4.1. *Norms with values in a power series ring.* Let us consider $U_1,\ldots,U_m$ formal variables, where $m \in \mathbb{N}^*$, and equip the set $\mathbb{R}_+[[U_1,\ldots,U_m]]$ with the product topology associated to the real topology on $\mathbb{R}^+$ and the natural identification $\mathbb{R}_+[[U_1,\ldots,U_m]] \simeq \mathbb{R}_+^{\mathbb{N}^m}$. Let us define a partial order on $\mathbb{R}_+[[U_1,\ldots,U_m]]$ by declaring that $\sum_{(n_1,\ldots,n_m)\in\mathbb{N}^m} s_{n_1,\ldots,n_m} U_1^{n_1}\ldots,U_m^{n_m} \leq \sum_{(n_1,\ldots,n_m)\in\mathbb{N}^m} s'_{n_1,\ldots,n_m} U_1^{n_1}\ldots U_m^{n_m}$ if, for all $(n_1,\ldots,n_m)\in\mathbb{N}^m$, we have $a_{n_1,\ldots,n_m} \leq a'_{n_1,\ldots,n_m}$. Let us also denote by $\times$ the usual multiplication of formal power series. If $S \leq S'$ in the sense above, then we have $S\times R \leq S'\times R$ for all power series R. Finally, the maximum of two elements $\sum_{(n_1,\ldots,n_m)\in\mathbb{N}^m} s_{n_1,\ldots,n_m} U_1^{n_1}\ldots,U_m^{n_m}$ and $\sum_{(n_1,\ldots,n_m)\in\mathbb{N}^m} s'_{n_1,\ldots,n_m} U_1^{n_1}\ldots,U_m^{n_m}$ is defined as $\sum_{(n_1,\ldots,n_m)\in\mathbb{N}^m} \max(s_{n_1,\ldots,n_m}, s'_{n_1,\ldots,n_m}) U_1^{n_1}\ldots,U_m^{n_m}$.

Let $\mathcal{C}$ be a K-algebra equipped with a map $\mathcal{N} : \mathcal{C} \to \mathbb{R}_+[[U_1,\ldots,U_m]]$ satisfying the axioms of an (ultrametric) algebra norm, with the notions of order (and maximum) on $\mathbb{R}_+[[U_1,\ldots,U_m]]$ defined above, and satisfying $\mathcal{N}(1_{\mathcal{C}}) = 1$. Then $\mathcal{C}$ is called a (ultrametric) normed K-algebra with norm $\mathcal{N}$.

Any (ultrametric) normed K-algebra in the sense of this definition is in particular a (ultra)metric space with the distance defined by the norm. The condition that $\mathcal{N}(1_{\mathcal{C}}) = 1$ is not systematically included in the usual definitions of normed algebras but we require it here to be satisfied for convenience. We are going to apply these definitions and notations to norms with values in $\mathbb{R}_+[[\Lambda]]$, $\mathbb{R}_+[[D]]$ and $\mathbb{R}_+[[\Lambda,D]]$, where the formal variable Λ, resp. D reflects the weight, resp. the depth on $\mathcal{O}^{ш,e_Z}$.

2.4.2. *$A\langle\langle e_Z\rangle\rangle$ as a ultrametric complete normed K-algebra.*

Notation 2.4.1. We recall that $\mathcal{W}(e_Z)$ is the set of words on e_Z. Let $(n,d)\in\mathbb{N}^2$; let $\mathcal{W}_n(e_Z)$, resp. $\mathcal{W}_{*,d}(e_Z)$, resp. $\mathcal{W}_{n,d}(e_Z) = \mathcal{W}_n(e_Z)\cap\mathcal{W}_{*,d}(e_Z)$ be the subset of $\mathcal{W}(e_Z)$ consisting of the words that are of weight n, resp. depth d, resp. weight n and depth d.

Definition 2.4.2. i) Let $\mathcal{N}_{\Lambda,D} : A\langle\langle e_Z\rangle\rangle \to \mathbb{R}_+[[\Lambda,D]]$ be the map

$$f \mapsto \mathcal{N}_{\Lambda,D}(f) = \sum_{(n,d)\in\mathbb{N}^2} \max_{w\in\mathcal{W}_{n,d}(e_Z)} \left|f[w]\right|_p \Lambda^n D^d$$

ii) Let $\mathcal{N}_\Lambda : A\langle\langle e_Z\rangle\rangle \longrightarrow \mathbb{R}_+[[\Lambda]]$ be the map

$$f \mapsto \mathcal{N}_\Lambda(f) = \sum_{n\in\mathbb{N}} \max_{w\in\mathcal{W}_n(e_Z)} \left|f[w]\right|_p \Lambda^n$$

Let $\mathbb{R}_+[[\Lambda,D]]'$ be the subset of $\mathbb{R}_+[[\Lambda,D]]$ made of elements such that the coefficient of $\Lambda^{n_1}D^{n_2}$ vanishes if $n_2 > n_1$, and let $\max_D : \mathbb{R}_+[[\Lambda,D]]' \mapsto \mathbb{R}_+[[\Lambda]]$ be the map

$$\sum_{(n_1,n_2)\in\mathbb{N}^2} a_{n_1,n_2}\Lambda^{n_1}D^{n_2} \mapsto \sum_{n_1\in\mathbb{N}} \big(\max_{0\leq n_2\leq n_1} a_{n_1,n_2}\big)\Lambda^{n_1}$$

We have

$$\mathcal{N}_\Lambda = \max_D \ \circ \ \mathcal{N}_{\Lambda,D}|_{\mathbb{R}_+[[\Lambda,D]]'}$$

Since the map $\max_D$ is an increasing function with respect to the partial orders on $\mathbb{R}_+[[\Lambda, D]]$ and $\mathbb{R}_+[[\Lambda]]$, we have the implication

$$\mathcal{N}_{\Lambda,D}(f) \leq \mathcal{N}_{\Lambda,D}(g) \Rightarrow \mathcal{N}_\Lambda(f) \leq \mathcal{N}_\Lambda(g)$$

and $\mathcal{N}_\Lambda$ inherits of most of the properties of $\mathcal{N}_{\Lambda,D}$. It is clear that :

Proposition 2.4.3. The K-algebra $A\langle\langle e_Z\rangle\rangle$ equipped with $\mathcal{N}_{\Lambda,D}$, resp. $\mathcal{N}_\Lambda$ is an ultrametric complete K-normed algebra, in the sense of §2.4.1 ; the topologies induced by $\mathcal{N}_{\Lambda,D}$ and $\mathcal{N}_\Lambda$ are both equal to the topology of pointwise convergence on $A\langle\langle e_Z\rangle\rangle = \mathrm{Map}(\mathcal{W}(e_Z), A)$.

Indeed, since, for all $n \in \mathbb{N}$, $\mathcal{W}_n(e_Z)$ is finite, the uniform convergence on each $\mathcal{W}_n(e_Z)$ is equivalent to simple convergence on $\mathcal{W}(e_Z)$.

2.4.3. *Bounded and summable elements of $A\langle\langle e_Z\rangle\rangle$.* There is another topology on $K\langle\langle e_Z\rangle\rangle$, that it is natural to use in order to deal with certain infinite summations of words of a fixed depth and with weight tending to ∞ ; if we restrict ourselves to a subset of elements satisfying certain conditions of bounds on their valuations, this topology is defined by a norm with values in $\mathbb{R}_+[[D]]$.

2.4.3.a. **Bounded elements**

Definition 2.4.4. Let $A\langle\langle e_Z\rangle\rangle_b$, be the subset of $A\langle\langle e_Z\rangle\rangle$ consisting in the elements f such that, for each $d \in \mathbb{N}^*$, we have

$$\sup_{w \in \mathcal{W}_{*,d}(e_Z)} |f[w]|_p < +\infty$$

Definition 2.4.5. i) Let $\mathcal{N}_D : A\langle\langle e_Z\rangle\rangle_b \to \mathbb{R}_+[[D]]$ be the map defined by :

$$\mathcal{N}_D(f) = \sum_{d\in\mathbb{N}} \left(\sup_{w\in\mathcal{W}_{*,d}(e_Z)} |f[w]|_p \right) D^d$$

ii) Let $\mathbb{R}_+[[\Lambda, D]]_b$ be the subset of $\mathbb{R}_+[[\Lambda, D]]$ consisting of the elements $\sum_{(n_1,n_2)\in\mathbb{N}^2} a_{n_1,n_2}\Lambda^{n_1}D^{n_2}$ such that, for all $n_2 \in \mathbb{N}$ we have $\sup_{n_1\in\mathbb{N}} a_{n_1,n_2} < +\infty$.

Let $\sup_\Lambda : \mathbb{R}_+[[\Lambda, D]]_b \mapsto \mathbb{R}_+[[D]]$ be the map

$$\sum_{(n_1,n_2)\in\mathbb{N}^2} a_{n_1,n_2}\Lambda^{n_1}D^{n_2} \mapsto \sum_{n_2\in\mathbb{N}} \big(\sup_{n_1\in\mathbb{N}} a_{n_1,n_2}\big) D^{n_2}$$

We have

$$\mathcal{N}_D = \sup_\Lambda \ \circ \ \mathcal{N}_{\Lambda,D}|_{K\langle\langle e_Z\rangle\rangle_b}$$

and since $\sup_\Lambda$ is an increasing function with respect to the partial orders on $\mathbb{R}_+[[\Lambda, D]]$ and $\mathbb{R}_+[[D]]$, we have the implication

$$\mathcal{N}_{\Lambda,D}(f) \leq \mathcal{N}_{\Lambda,D}(g) \Rightarrow \mathcal{N}_D(f) \leq \mathcal{N}_D(f)$$

and $\mathcal{N}_D$ inherits, like $\mathcal{N}_\Lambda$, of most of the properties of $\mathcal{N}_{\Lambda,D}$. In this case it is also clear that :

Proposition 2.4.6. $A\langle\langle e_Z\rangle\rangle_b$ equipped with $\mathcal{N}_D$ is a complete ultrametric normed K-algebra in the sense of §2.4.1 ; its topology is the topology on $A\langle\langle e_Z\rangle\rangle_b \subset \mathrm{Map}(\mathcal{W}(e_Z), A)$ of uniform convergence on all the subsets $\mathcal{W}_{*,d}(e_Z)$, $d \in \mathbb{N}^*$.

2.4.3.b. **Summable elements**

Definition 2.4.7. Let $A\langle\langle e_Z\rangle\rangle_\Sigma$, be the subset of $A\langle\langle e_Z\rangle\rangle$ consisting of elements f such that, for all $d \in \mathbb{N}^*$, we have :

$$\sup_{w\in\mathcal{W}_{n,d}(e_Z)} \big|f[w]\big|_p \underset{n\to+\infty}{\longrightarrow} 0$$

By the ultrametricity of K, this is also the set of elements f such that, for every sequence $(w_l)_{l\in\mathbb{N}}$ of $\mathcal{W}(e_Z)$ satisfying $\limsup \mathrm{depth}(w_l) < +\infty$ and $\mathrm{weight}(w_l) \to +\infty$, we have

$$\sum_{l\in\mathbb{N}} |f[w_l]|_p < +\infty$$

The condition defining $A\langle\langle e_Z\rangle\rangle_\Sigma$ can also be formulated by saying that, for each $d \in \mathbb{N}$, the coefficient of D^d in $\mathcal{N}_{\Lambda,D}(f)$, viewed as a function of Λ, defines a rigid analytic function on $\mathcal{O}_A$; perhaps surprisingly, this way to formulate the definition will actually become natural in some next parts of the theory.

Proposition 2.4.8. $A\langle\langle e_Z\rangle\rangle_\Sigma$ equipped with $\mathcal{N}_D$ is a complete ultrametric normed K-algebra in the sense of §2.4.1.

Proof. Standard arguments of general topology. □

2.4.4. *The subset* $\Pi_{z,0}(A)$. By the inclusion $\Pi_{z,0}(A) \subset A\langle\langle e_Z\rangle\rangle$, $\Pi_{z,0}(A)$ resp. $\Pi_{z,0}(A)\cap A\langle\langle e_Z\rangle\rangle_b$ is equipped with the $\mathcal{N}_{\Lambda,D}$-topology resp. the $\mathcal{N}_D$-topology. From now on, we are going to obtain properties that are specific to grouplike series on e_Z.

2.4.4.a. **Generalities**

Proposition 2.4.9. The natural isomorphisms $\mathrm{Diag}^\xi\left(\prod_{i=1}^N \Pi_{z_i,0}\right)(A) \simeq \Pi_{z,0}(A)$, $z \in \{z_1,\ldots,z_N\}$, are homeomorphisms for the $\mathcal{N}_{\Lambda,D}$-topology, as are their restrictions $\mathrm{Diag}^\xi\left(\prod_{i=1}^N \tilde{\Pi}_{z_i,0}\right)(A) \simeq \tilde{\Pi}_{z,0}(A)$.

Proof. Clear. □

Lemma 2.4.10. Let $f \in \Pi_{z,0}(A)$, and let $l \in \mathbb{N}^*$. We have :

$$\mathcal{N}_{\Lambda,D}(f^l) = \mathcal{N}_{\Lambda,D}(f)$$

Proof. We must prove that for all $n, d \in \mathbb{N}^*$, we have :

$$\max_{\substack{(w_1,\ldots,w_l)\in\mathcal{W}(e_Z)^l \\ \text{s.t. } w_1\ldots w_l\in\mathcal{W}_{n,d}(e_Z)}} \Big|\prod_{i=1}^l f[w_i]\Big|_p = \max_{w\in\mathcal{W}_{n,d}(e_Z)} \big|f[w]\big|_p$$

The inequality $\geq$ is obtained by choosing $w_2 = \ldots = w_l = \emptyset$ in the left hand side since $f[\emptyset] = 1$; the inequality $\leq$ follows from the shuffle equation for f and from that the shuffle product restricts, for all $n_1, n_2, d_1, d_2 \in \mathbb{N}$, to a map $ш : \mathcal{W}_{n_1,d_1}(e_Z) \times \mathcal{W}_{n_2,d_2}(e_Z) \to \mathbb{Z}[\mathcal{W}_{n_1+n_2,d_1+d_2}(e_Z)]$. □

Proposition 2.4.11. We have, for all $f \in \Pi_{z,0}(A)$:

$$\mathcal{N}_{\Lambda,D}(f^{-1}) = \mathcal{N}_{\Lambda,D}(f)$$

Proof. It is sufficient to prove that $\mathcal{N}_{\Lambda,D}(f^{-1}) \leq \mathcal{N}_{\Lambda,D}(f)$. We have $f^{-1} = \sum_{l\in\mathbb{N}}(1-f)^l$, where for each $w \in \mathcal{O}^{ш,e_Z}$, the sum $\sum_{l\in\mathbb{N}}(1-f)^l[w]$ is finite. In particular, the ultrametric triangle inequality for $\mathcal{N}_{\Lambda,D}$ has a sense and remains true for this infinite sum, and we have $\mathcal{N}_{\Lambda,D}(f^{-1}) \leq \max_{l\in\mathbb{N}} \mathcal{N}_{\Lambda,D}((1-f)^l) \leq \max_{l\in\mathbb{N}} \mathcal{N}_{\Lambda,D}(f^l) = \mathcal{N}_{\Lambda,D}(f)$, where the last equality follows from Lemma 2.4.10 and the last inequality follows from the binomial expansion of $(1-f)^l$ and again from the ultrametric triangle inequality for $\mathcal{N}_{\Lambda,D}$. □

Proposition 2.4.12. We have, for all $f \in \Pi_{z,0}(A)$,

$$\mathcal{N}_{\Lambda,D}(\mathrm{Ad}_f(e_z)) \leq \Lambda D \mathcal{N}_{\Lambda,D}(f)$$

Proof. Follows from Proposition 2.3.6. □

Proposition 2.4.13. We have, for all $f, g \in \Pi_{z,0}(A)$:

$$\mathcal{N}_{\Lambda,D}(g \circ^{\mathrm{DR}} f) \leq \mathcal{N}_{\Lambda,D}(g) \times \mathcal{N}_{\Lambda,D}(f)$$
$$\mathcal{N}_{\Lambda,D}(\tau(\lambda)(f))(\Lambda, D) = \mathcal{N}_{\Lambda,D}(f)(\lambda\Lambda, D)$$

Moreover, the Ihara product $\circ^{\mathrm{DR}} : \Pi_{z,0}(A) \times \Pi_{z,0}(A) \to \Pi_{z,0}(A)$ and $\tau : A^* \times \Pi_{z,0}(A) \to \Pi_{z,0}(A)$ are continuous relatively to the $\mathcal{N}_{\Lambda,D}$-topology on $\Pi_{z,0}(A)$ and the topology on A.

Proof. The inequality follows from Corollary 2.3.7 and the equality is clear. The assertions on continuity are also clear. □

Remark 2.4.14. All statements from Lemma 2.4.10 to Proposition 2.4.13 remain true with $\mathcal{N}_\Lambda$ instead of $\mathcal{N}_{\Lambda,D}$, except for the last assertion on τ, which is replaced by

$$\mathcal{N}_\Lambda(\tau(\lambda)(f))(\Lambda) = \mathcal{N}_\Lambda(f)(\lambda\Lambda)$$

2.4.4.b. **Bounded and summable elements**

Definition 2.4.15. Let $\Pi_{z,0}(A)_b = \Pi_{z,0}(A) \cap A\langle\langle e_Z \rangle\rangle_b$ and $\Pi_{z,0}(A)_\Sigma = \Pi_{z,0}(A) \cap A\langle\langle e_Z \rangle\rangle_\Sigma$.

They are equipped with the restriction of the $\mathcal{N}_{\Lambda,D}$-topology and of the $\mathcal{N}_D$-topology.

Lemma 2.4.16. $\Pi_{z,0}(A)_\Sigma$ and $\Pi_{z,0}(A)_b$ are subgroups of $\Pi_{z,0}(A)$ for the usual multiplication of grouplike series.

Proof. Standard arguments of general topology and elementary analysis. □

Lemma 2.4.17. The adjoint action $\mathrm{Ad}(e_z)$, induces maps $\Pi_{z,0}(A)_b \to K\langle\langle e_Z \rangle\rangle_b$ and $\Pi_{z,0}(A)_\Sigma \to K\langle\langle e_Z \rangle\rangle_\Sigma$.

Proof. Follows from Proposition 2.4.12. □

Proposition 2.4.18. i) $\Pi_{z,0}(A)_b$, $\Pi_{z,0}(A)_\Sigma$ and their images by $\mathrm{Ad}(e_Z)$ are stable by $\tau(\lambda)$ for $\lambda \in A^\times$ such that $|\lambda|_A \leq 1$.
ii) $\Pi_{z,0}(A)_\Sigma$, $\Pi_{z,0}(A)_b$ are subgroups of $\Pi_{z,0}(A)$ for $\circ^{\mathrm{DR}}$; their images by $\mathrm{Ad}(e_z)$ are subgroups for $\circ^{\mathrm{DR}}_{\mathrm{Ad}}$.

Proof. Follows from Proposition 2.4.13. □

Proposition 2.4.19. Equipped with the $\mathcal{N}_D$-topology, $\Pi_{z,0}(A)_b$ is closed inside $A\langle\langle e_Z \rangle\rangle_b$, and $\Pi_{z,0}(A)_\Sigma$ is closed inside $\Pi_{z,0}(A)_b$; in particular, $\Pi_{z,0}(A)_b$ and $\Pi_{z,0}(A)_\Sigma$ are complete.

Proof. Standard arguments of general topology. □

2.4.5. *Application to the analytic sections on $\mathbb{P}^{1,an} -]z_1, \ldots, z_{r+1}[$ of the torsor of paths starting at $\vec{1}_0$.* For our applications, most of the time, A will be either K or a ring of polynomials over K in one variable, but there exists another situation : let us consider the set of rigid analytic sections on $\mathbb{P}^{1,\mathrm{an}} - \cup_{z \in Z - \{0,\infty\}}]z[$ of the fundamental torsor $\pi_1^{\mathrm{un,DR}}(X_K, \vec{1}_0, *)$ of paths starting at $\vec{1}_0$; it is trivialized at $\vec{1}_0$, and thus it is identified with

$$\Pi_{0,0}\big(\mathfrak{A}(\mathbb{P}^1 - \cup_{z \in Z - \{0,\infty\}}]z[)\big)$$

The space of analytic functions $\mathfrak{A}(\mathbb{P}^1 - \cup_{z\in Z-\{0,\infty\}}]z[)$ is a complete normed K-algebra, where its topology is defined by the uniform norm of rigid analytic functions on $\mathbb{P}^1 - \cup_{z\in Z-\{0,\infty\}}]z[$; but it is also isomorphic, as a topological K-algebra, to $\text{comp}_{U^{\text{an}}}\ \mathfrak{A}(\mathbb{P}^1 - \cup_{z\in Z-\{0,\infty\}}]z[) \subset K^{\mathbb{N}}$ (I-1 §4) that consists of the sequences of coefficients $(c_n)_{n\in\mathbb{N}}$ of the series expansions $\sum_{n\in\mathbb{N}} c_n z^n$ at 0 of elements of $\mathfrak{A}(\mathbb{P}^1 - \cup_{z\in Z-\{0,\infty\}}]z[)$, equipped with the uniform norm of $K^{\mathbb{N}}$.

Convention 2.4.20. When we will apply the previous definitions to $A = \mathfrak{A}(\mathbb{P}^1 - \cup_{z\in Z-\{0,\infty\}}]z[)\big)$, we will use the uniform norm "$\sup_{n\in\mathbb{N}} |c_n|_p$" applied to sequences of coefficients of the series expansion at 0 of elements of A.

3. Proofs in the De Rham setting

In §3.1, we translate the equation of horizontality on the level of the coefficients of the series expansion at 0 of elements of $A_{\text{Col}}(\mathbb{P}^1 - \{z_1, \ldots, z_N\})$. In §3.2, we show that, in a certain limit, the terms that involve $\text{Li}^{\dagger}_{p,\alpha}$ of the equation of horizontality rewritten as in §3.1 tend to 0. In order to treat the similar limit of the remaining terms, in §3.3, we define and prove the basic properties of the De Rham-rational harmonic Ihara action. The limit of the remaining terms is expressed in terms of it in §3.4. We conclude in §3.5 the proof of the De Rham part of Theorem I-2.a.

3.1. **Coefficients of the series expansion at** 0 **of the horizontality equation.** If $S \in K[[z]][\log(z)]$, we denote the coefficient in S of $z^n \log(z)^m$, for $n, m \in \mathbb{N}$, by $S[z^n \log(z)^m]$. Let us recall how weighted multiple harmonic sums are related to the coefficients of the series expansion of Li_p^{KZ} at 0, and let us extend the notion of weighted multiple harmonic sums to words whose furthest to the right letter is e_0. We fix $l \in \mathbb{N}^*$.

Lemma 3.1.1. Let $d \in \mathbb{N}^*$, $s_d, \ldots, s_1 \in (\mathbb{N}^*)^{d+1}$, $i_{d+1}, \ldots, i_1 \in \{1, \ldots, N\}$. Let $w = \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \right)$ and $w^{(p^\alpha)} = \left(\begin{array}{c} z^{p^\alpha}_{i_{d+1}}, \ldots, z^{p^\alpha}_{i_1} \\ s_d, \ldots, s_1 \end{array} \right)$. Let $w_l = e_0^{l-1} e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}}$. For all $n \in \mathbb{N}^*$, we have :

$$\tau(n)\,\text{Li}^{\text{KZ}}_{p,X_K}[w_l][z^n] = (-1)^{d+1}\,\text{har}_n\big(w\big)$$
$$\tau(n)\,\text{Li}^{\text{KZ}}_{p,X_K^{(p^\alpha)}}[w_l][z^n] = (-1)^{d+1}\,\text{har}_n\big(w^{(p^\alpha)}\big)$$

Let us now fix $r \in \mathbb{N}$. We have $\text{Li}^{\text{KZ}}_{p,X_K}[e_0](z) = \text{Li}^{\text{KZ}}_{p,X_K^{(p^\alpha)}}[e_0](z) = \log_p(z)$, thus, with the same notations :

$$\tau(n)\,\text{Li}^{\text{KZ}}_{p,X_K}(z)[w_l e_0^r] = \sum_{\substack{r', r_{d+1}, \ldots, r_1 \geq 0 \\ r_{d+1}+\ldots+r_1+r'=r}} \Bigg[$$

$$\binom{-l}{r_{d+1}} \prod_{i=1}^d \binom{-s_i}{r_i} \tau(n)\,\text{Li}^{\text{KZ}}_{p,X_K}(z)[e_0^{l+r_{d+1}-1} e_{z_{i_{d+1}}} e_0^{s_d+r_d-1} e_{z_{i_d}} \ldots e_0^{s_1+r_1-1} e_{z_{i_1}}] \frac{(n \log_p(z))^{r'}}{r'!} \Bigg]$$

and similarly for $\tau(n)\,\text{Li}^{\text{KZ}}_{p,X_K^{(p^\alpha)}}$. It is convenient for our purposes to extend the notion of weighted multiple harmonic sums as follows :

Definition 3.1.2. Let l_f be a formal variable and

$$\text{har}_n \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ s_d, \ldots, s_1; r \end{array} \right)(l_f) = \sum_{\substack{r_{d+1}, \ldots, r_1 \geq 0 \\ r_{d+1}+\ldots+r_1=r}} \binom{-l_f}{r_{d+1}} \prod_{i=1}^d \binom{-s_i}{r_i} \text{har}_n \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_1 \\ s_d + r_d, \ldots, s_1 + r_1 \end{array} \right)$$

in $\mathbb{Q}(\xi)[l_f]$; in other terms :

Lemma 3.1.3. We have, with $w_r = \left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1;r \end{array}\right)$ and $w_r^{(p^\alpha)} = \left(\begin{array}{c} z_{i_{d+1}}^{p^\alpha},\ldots,z_{i_1}^{p^\alpha} \\ s_d,\ldots,s_1;r \end{array}\right)$:

$$\tau(n)\,\mathrm{Li}^{\mathrm{KZ}}_{p,X_K}[w_{l,r}][z^n\log(z)^0] = \mathrm{har}_n(w_r)(l)$$
$$\tau(n)\,\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}[w_{l,r}][z^n\log(z)^0] = \mathrm{har}_n(w_r^{(p^\alpha)})(l)$$

Now, we recall from I-1 (Lemma 3.4.1) the following simple but crucial fact.

Lemma 3.1.4. For all $l\in\mathbb{N}^*$, we have $\mathrm{Li}^\dagger_{p,\alpha}[e_0^l](z)=0$.

One also has $\mathrm{Li}^\dagger_{p,\alpha}[w][z^0] = \mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}[w][z^0\log(z)^0] = \mathrm{Li}^{\mathrm{KZ}}_{p,X_K}[w][z^0\log(z)^0]=1$.
Since the horizontality equation is valid for any branch of the p-adic logarithm, let us choose the one such that $\log_p(p)=0$.

Lemma 3.1.5. Let $f,g\in K[[z]][\log_p(z)]$ such that, for all $\{z\in K \mid 0<|z|_p<1\}$, we have $f(z)=g(z)$. Then, for all $n\in\mathbb{N}$, we have $f[z^n\log(z)^0]=g[z^n\log(z)^0]$.

This is proved by specializing z to powers of p. One can now translate the horizontality equation (2.1.2) as follows on the coefficients $[z^n\log(z)^0]$.

Proposition 3.1.6. With the same notations we have :

$$(3.1.1)\quad \tau(n)\Bigg[\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}(z)\big(e_0,(\Phi^{(z_1)}_{p,\alpha})^{-1}e_{z_1}\Phi^{(z_1)}_{p,\alpha},\ldots,(\Phi^{(z_N)}_{p,\alpha})^{-1}e_{z_1}\Phi^{(z_N)}_{p,\alpha}\big)[w_{l,r}][z^n\log(z)^0]+$$
$$\mathrm{Li}^\dagger_{p,\alpha}[w_{l,r}][z^{p^\alpha n}]+$$
$$\sum_{*}\mathrm{Li}_{p,\alpha}[u_{l,r}][z^j].\,\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}(z^{p^{|m|}})\big(e_0,(\Phi^{(z_1)}_{p,\alpha})^{-1}e_{z_1}\Phi^{(z_1)}_{p,\alpha},\ldots,(\Phi^{(z_N)}_{p,\alpha})^{-1}e_{z_1}\Phi^{(z_N)}_{p,\alpha}\big)[v_r][z^{n-\frac{j}{p^\alpha}}\log(z)^0]\Bigg]$$
$$=(-1)^{d+1}\,\mathrm{har}_{p^\alpha n}(w_r)(l)$$

where $*=\{1\le j\le p^\alpha n-1 \mid p^\alpha|j\}\times\{(u_{l,r},v_r) \mid w_{l,r}=u_{l,r}v_r,\ \mathrm{depth}(u_l)\ge 1,\ v\neq\emptyset\}$.

Proof. We take, from the horizontality equation (2.1.2), the coefficient $[w_l]$, and then, its coefficient $[z^{p^\alpha n}]$ in the series expansion at 0 with respect to z ; and we apply $\tau(n)$, defined in §2.1.2, to the equality. We apply finally the two previous lemmas. The sum over j arising from the product of two power series in z runs over $\{0,\ldots,p^\alpha n\}$, and the sum over couples $(u_{l,r},v_r)$ runs over all the possible deconcatenations of the word $w_{l,r}$: $w_{l,r}=u_{l,r}v_r$. We use that, for any power series S, we have $S(z^{p^\alpha})[z^{p^\alpha n}]=S(z)[z^n]$. By the vanishing properties of the previous lemma, the sum over j restricts to $\alpha\in\{1,\ldots,p^\alpha n-1\}$, and the sum over $u_{l,r}$ restricts to terms such that $\mathrm{depth}(u_{l,r})\ge 1$. □

From now on we are interested in the limit when $l\to\infty$ of equation (3.1.1).

3.2. **Limit of the terms of equation (3.1.1) involving $\mathrm{Li}^\dagger_{p,\alpha}$ when $l\to\infty$.** The second ingredient of the proof is the lower bounds of valuations of $\mathrm{Li}^\dagger_{p,\alpha}$ of Appendix to Theorem I-1 ; we reproduce here only a rough version of them, that is enough for the present proof, formulated in terms of the norms introduced in §2.4.

Lemma 3.2.1. i) In the sense of Convention 2.4.20, we have :

$$(3.2.1)\qquad \mathcal{N}_{\Lambda,D}(\mathrm{Li}^\dagger_{p,\alpha})\le\sum_{n,d\ge 1}p^{-\inf_{l\in\mathbb{N}}\big(l+n-(2d\log(2d)-d+1)-2d\log(l+n)\big)}\Lambda^n D^d$$

whence

$$\mathrm{Li}^{\dagger}_{p,\alpha} \in \mathfrak{A}(\mathbb{P}^1 - \cup_{z \in Z - \{0,\infty\}}]z[)_{\Sigma}$$

ii) For all $i \in \{1,\ldots,N\}$, we have :

$$(3.2.2) \qquad \mathcal{N}_{\Lambda,D}(\Phi^{(z_i)}_{p,\alpha}) \leq \sum_{n,d \geq 1} p^{-\big(n - \big(2d\log(2dn) - d + 1\big)\big)} \Lambda^n D^d$$

whence

$$\Phi^{(z_i)}_{p,\alpha} \in \Pi_{z_i,0}(K)_{\Sigma}$$

iii) For all $i \in \{1,\ldots,N\}$, we have

$$(3.2.3) \qquad \mathcal{N}_{\Lambda,D}(\mathrm{Ad}_{\Phi^{(z)}_{p,\alpha}}(e_z)) \leq \sum_{n,d \geq 1} p^{-\big(n - \big(2d\log(2dn) - d + 1\big)\big)} \Lambda^{n+1} D^{d+1}$$

whence

$$\mathrm{Ad}_{\Phi^{(z_i)}_{p,\alpha}}(e_{z_i}) \in \big(\mathrm{Ad}_{\Pi_{z_i,0}(K)}(e_{z_i})\big)_{\Sigma}$$

Proof. The bounds of values of $\mathcal{N}_{\Lambda,D}$ follow from Appendix to Theorem I-1 and Proposition 2.4.12. They imply the rest of the statement because the function $l \mapsto l + n - 1 - 2d\log(l+n)$ is increasing on the interval $[2d-n,\infty[$, thus, for $n > 2d$, the inf of the formula (3.2.1) is $\leq p^{-\big(n-1-2d\log\big(2dn\big)\big)}$, which is also equal to the bound of (3.2.2), and tends to 0 when $n \to +\infty$ and d is fixed. □

The coefficients $\mathrm{har}^{\dagger_{p,\alpha}}_n$ of the series expansion of of $\mathrm{Li}^{\dagger}_{p,\alpha}$ at 0, being a regularization of the coefficients har_n, must be significantly smaller p-adically than har_n. We prove now that this property of smallness implies a property of vanishing in a certain limit.

Lemma 3.2.2. The following term in Proposition 3.1.6 (where Σ_* is the summation described there) :

$$(3.2.4) \quad \mathrm{Li}^{\dagger}_{p,\alpha}[w_{l,r}][z^{p^{\alpha}n}] +$$

$$\sum_{*} \mathrm{Li}_{p,\alpha}[u_{l,r}][z^j].\,\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}(z^{p^{|m|}})\big(e_0, (\Phi^{(z_1)}_{p,\alpha})^{-1} e_{z_1} \Phi^{(z_1)}_{p,\alpha}, \ldots, (\Phi^{(z_N)}_{p,\alpha})^{-1} e_{z_1} \Phi^{(z_N)}_{p,\alpha}\big)[v_r][z^{n-\frac{j}{p^\alpha}}\log(z)^0]\Bigg]$$

tends to 0 when $l \to +\infty$.

Proof. The indexation of v_r does not depend on l ; thus, the factors depending on v_r in the second line are contained in a bounded subset of K depending only on $\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1;r \end{array}\right)$. Moreover, each $u_{l,r}$ is determined by the corresponding v_r, and there are a finite number of such v_r's. Finally, we have $\limsup \mathrm{depth}\, u_{l,r} < +\infty$ and $\mathrm{weight}\, u_{l,r} \to +\infty$, and similarly for $w_{l,r}$. Whence the result by Lemma 3.2.1. □

3.3. **The De Rham-rational harmonic Ihara action.** As in §2.4, let us consider A a complete normed K-algebra.

3.3.1. *Adjoint Ihara action and its extension to $A\langle\langle e_Z\rangle\rangle$.*

Definition 3.3.1. i) We call adjoint Ihara action the map

$$\circ^{\mathrm{DR}}_{\mathrm{Ad}} : \mathrm{Ad}_{\Pi_{z,0}(K)}(e_z) \times \mathrm{Ad}_{\Pi_{z,0}(K)}(e_z) \to \mathrm{Ad}_{\Pi_{z,0}(K)}(e_z)$$

$$(h_z, f_z) \mapsto h_z \circ^{\mathrm{DR}}_{\mathrm{Ad}} f_z = f_z(e_0, h_{z_1}, \ldots, h_{z_N})$$

where $(h_{z_1},\dots,h_{z_N})$ is the preimage in $\mathrm{Diag}^{\xi}(\prod_{i=1}^{N}\mathrm{Ad}_{\Pi_{z_i,0}(K)}(e_{z_i}))$ of h_z.
ii) We call also adjoint Ihara action and denote by $\circ_{\mathrm{Ad}}^{\mathrm{DR}}$ the map $A\langle\langle e_Z\rangle\rangle^N \times A\langle\langle e_Z\rangle\rangle \to A\langle\langle e_Z\rangle\rangle$ that sends

$$\big((h_{z_1},\dots,h_{z_N}),f\big) \mapsto (h_{z_1},\dots,h_{z_N}) \circ_{\mathrm{Ad}}^{\mathrm{DR}} f = f(e_0,h_{z_1},\dots,h_{z_N})$$

The map $\circ_{\mathrm{Ad}}^{\mathrm{DR}}$ in i) above is characterized by the commutativity of the following diagram :

$$\begin{array}{ccc} \Pi_{z,0}(K)\times\Pi_{z,0}(K) & \overset{\circ^{\mathrm{DR}}}{\longrightarrow} & \Pi_{z,0}(K) \\ \downarrow_{\mathrm{Ad}(e_z)\times\mathrm{Ad}(e_z)} & & \downarrow_{\mathrm{Ad}(e_z)} \\ \mathrm{Ad}_{\Pi_{z,0}(K)}(e_z)\times\mathrm{Ad}_{\Pi_{z,0}(K)}(e_z) & \overset{\circ^{\mathrm{DR}}_{\mathrm{Ad}}}{\longrightarrow} & \mathrm{Ad}_{\Pi_{z,0}(K)}(e_z) \end{array}$$

In particular, it is a group law on $\mathrm{Ad}_{\Pi_{z,0}(K)}(e_z)$. More generally, since the composition of formal power series is associative, we have in particular :

Lemma 3.3.2. The adjoint Ihara action extended as in ii) above is associative.

This justifies the terminology in ii) above.

3.3.2. *Definition and continuity of the harmonic Ihara action.*

Definition 3.3.3. i) Let $A\langle\langle e_Z\rangle\rangle^{\mathrm{const}} \subset A\langle\langle e_Z\rangle\rangle$ be the vector subspace consisting of the elements $f\in A\langle\langle e_Z\rangle\rangle$ such that, for all words w on e_Z, the sequence $(f[e_0^l w])_{l\in\mathbb{N}}$ is constant.
ii) Let $A\langle\langle e_Z\rangle\rangle^{\lim} \subset A\langle\langle e_Z\rangle\rangle$ be the vector subspace consisting of the elements $f\in A\langle\langle e_Z\rangle\rangle$ such that, for all words w on e_Z, the sequence $(f[e_0^l w])_{l\in\mathbb{N}}$ has a limit in A when $l\to\infty$.
iii) We have a map that we call "limit" :

$$\lim : A\langle\langle e_Z\rangle\rangle^{\lim} \to A\langle\langle e_Z\rangle\rangle^{\mathrm{const}}$$

defined by, for all words w,

$$(\lim f)[w] = \lim_{l\to\infty} f[e_0^l w]$$

Lemma 3.3.4. The map lim of Definition 3.3.3 is continuous for restriction of the $\mathcal{N}_D$-topology on the source and target.

Proof. Clear. □

Proposition-Definition 3.3.5. The map :

$$\circ_{\mathrm{har}}^{\mathrm{DR,RT}} : \mathrm{Diag}^{\xi}\big(\prod_{i=1}^{N}\tilde{\Pi}_{z_i,0}(A)_{\Sigma}\big) \times \mathrm{Map}(\mathbb{N}, A\langle\langle e_Z\rangle\rangle^{\mathrm{const}}) \to \mathrm{Map}(\mathbb{N}, A\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$$

defined by the equation

$$(g_{z_1},\dots,g_{z_N}) \circ_{\mathrm{har}}^{\mathrm{DR,RT}} (n\mapsto f_n) = \big(n \mapsto \lim\big(\tau(n)(g_{z_1},\dots,g_{z_N}) \circ_{\mathrm{Ad}}^{\mathrm{DR}} f_n\big)\big)$$

is well-defined. We call it the *De Rham-rational harmonic Ihara action* of X_K.

Proof. Let $f\in A\langle\langle e_Z\rangle\rangle^{\mathrm{const}}$ and $(g_{z_1},\dots,g_{z_N}) \in \mathrm{Diag}^{\xi}\big(\prod_{i=1}^{N}\tilde{\Pi}_{z_i,0}(A)_{\Sigma}\big)$; let us take a sequence of words $(w_l)_{l\in\mathbb{N}}$ of the form $(e_0^l e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^r)_{l\in\mathbb{N}}$, with $i_1,\dots,i_{d+1}\in\{1,\dots,N\}$, $r\in\mathbb{N}$, we must show that $f(e_0,g_{z_1},\dots,g_{z_N})[w_l]$ has a limit when $l\to\infty$. The Proposition 2.3.5 gives a formula for $f(e_0,g_{z_1},\dots,g_{z_N})[w_l]$; it is easy to see that, because of the assumption on f, the formula depends on l in the following way :

$$f(e_0,g_{z_1},\dots,g_{z_N})[w_l] = \rho + \sum_{i=1}^{N}\sum_{u}\theta_{u,i}\sum_{b=1}^{l} g_{z_i}[e_0^{b-1}u] \tag{3.3.1}$$

where u runs over certain connected subsequences of $e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^r$, so does not depend on l, and $\rho, \theta_{u,i} \in K$ do not depend on l. (Indeed, the formula is a sum indexed by couples $((sw_j)_{j\in J}, C)$ where $(sw_j)_{j\in J}$ is a connected partition of an element $sw \in \mathcal{SW}(w)$, and C is a coloring of $(sw_j)_{j\in J}$; each $sw \in \mathcal{SW}(w_l)$ that is not maximally at the left of w (in the sense of Definition 2.3.1) gives a term in the sum that does not depend on l ; each other $sw \in \mathcal{SW}(w)$ gives a contribution that depends on l via the integer $b \in \{1,\dots,l\}$ such that sw_{i_1}, the furthest to the left element of a connected partition of sw, is of the form $e_0^b e_{z_{i_{d+1}}} sw'_{i_1}$.)
Because of the assumption of g, the right hand side of equation (3.3.1) converges when $l \to \infty$. □

If $f \in A\langle\langle e_Z\rangle\rangle^{\mathrm{const}}$ and $w = e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}}$, with $i_1,\dots,i_{d+1} \in \{1,\dots,N\}$, $s_1,\dots,s_d \in \mathbb{N}^*$, then we will denote by $f[w] = f\Big(\begin{array}{c} z_{i_{d+1}},\dots,z_{i_1} \\ s_d,\dots,s_1 \end{array}\Big)$; as usual the sequence $(z_{i_{d+1}},\dots,z_{i_1})$ can be omitted in the notation in the case of $\mathbb{P}^1 - \{0,1,\infty\}$.

Examples 3.3.6. Let $f = (f_n)_{n\in\mathbb{N}}$. For $\mathbb{P}^1 - \{0,1,\infty\}$, we have :
i) $d = 1$: for all $s_1 \in \mathbb{N}^*$,

$$(g \circ_{\mathrm{har}}^{\mathrm{DR,RT}} f)(s_1) = \Big(f_n(s_1) + \sum_{b\in\mathbb{N}} n^{s_1+b} \operatorname{Ad}_g(e_1)[e_0^b e_1 e_0^{s_1-1} e_1]\Big)_{n\in\mathbb{N}} \tag{3.3.2}$$

ii) $d = 2$: for all $s_1, s_2 \in \mathbb{N}^*$,

$$(g \circ_{\mathrm{har}}^{\mathrm{DR,RT}} f)(s_2,s_1) = \Bigg(f_n(s_2,s_1) + \sum_{b\in\mathbb{N}} n^{b+s_2+s_1} \operatorname{Ad}_g(e_1)[e_0^b e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] \tag{3.3.3}$$
$$+ \sum_{r_2=0}^{s_2-1} f_n(s_2-r_2) n^{r_2+s_1} \operatorname{Ad}_g(e_1)[e_0^{r_2} e_1 e_0^{s_1-1} e_1] + \sum_{r_1=0}^{s_1-1} f_n(s_1-r_1) \sum_{b\in\mathbb{N}} n^{b+s_2+r_1} \operatorname{Ad}_g(e_1)[e_0^b e_1 e_0^{s_2-1} e_1 e_0^{r_1}]\Bigg)_{n\in\mathbb{N}}$$

Proposition 3.3.7. The De Rham-rational harmonic Ihara action is continuous for the $\mathcal{N}_D$-topology on $\mathrm{Diag}^{\xi}\big(\prod_{i=1}^N \tilde{\Pi}_{z_i,0}(A)_\Sigma\big) \simeq \tilde{\Pi}_{z,0}(A)_\Sigma$, and the product topology on $\mathrm{Map}(\mathbb{N}, A\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$ indexed by $\mathbb{N}$ of the $\mathcal{N}_D$-topologies.

Proof. By Proposition 2.4.13, $\circ_{\mathrm{Ad}}^{\mathrm{DR}}$ is continuous, and by Lemma 3.3.4, lim is continuous. □

3.3.3. *Algebraic properties of the De Rham-rational harmonic Ihara action.*

Proposition 3.3.8. The De Rham rational harmonic Ihara action is a group action of the group $(\tilde{\Pi}_{z,0}(A)_\Sigma, \circ_{\mathrm{Ad}}^{\mathrm{DR}})$ on $\mathrm{Map}(\mathbb{N}, A\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$.

Proof. Let $(f_n)_{n\in\mathbb{N}} \in \mathrm{Map}(\mathbb{N}, A\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$, and let $g_1, g_2 \in \tilde{\Pi}_{z,0}(A)_\Sigma$. By the associativity of $\circ_{\mathrm{Ad}}^{\mathrm{DR}}$ extended to the whole of $A\langle\langle e_Z\rangle\rangle$ (Proposition 3.3.2), we have, for all $n \in \mathbb{N}$:

$$\tau(n)(g_2) \circ_{\mathrm{Ad}}^{\mathrm{DR}} (\tau(n)(g_1) \circ_{\mathrm{Ad}}^{\mathrm{DR}} f_n) = (\tau(n)(g_2) \circ_{\mathrm{Ad}}^{\mathrm{DR}} \tau(n)(g_1)) \circ_{\mathrm{Ad}}^{\mathrm{DR}} f_n \tag{3.3.4}$$

By Proposition-Definition 3.3.5, by $\tau(n)(g_2) \circ_{\mathrm{Ad}}^{\mathrm{DR}} \tau(n)(g_1) = \tau(n)(g_2 \circ_{\mathrm{Ad}}^{\mathrm{DR}} g_1)$ and by Proposition 2.4.18, the right hand side of (3.3.4) is in $A\langle\langle e_Z\rangle\rangle^{\mathrm{lim}}$ and its limit is the n-th term of the sequence $(g_2 \circ_{\mathrm{Ad}} g_1) \circ_{\mathrm{har}}^{\mathrm{DR,RT}} (f_n)_{n\in\mathbb{N}}$. The following lemma shows that the expression $g_2 \circ_{\mathrm{har}}^{\mathrm{DR,RT}} \big(g_1 \circ_{\mathrm{har}}^{\mathrm{DR,RT}} (f_n)_{n\in\mathbb{N}}\big)$ is well-defined and equal to the sequence indexed by $n \in \mathbb{N}$ of limits of the left hand-side of (3.3.4). □

Lemma 3.3.9. Let $f' \in A\langle\langle e_Z\rangle\rangle^{\mathrm{const}}$ and $g'_1, g'_2 \in \tilde{\Pi}_{z,0}(A)_\Sigma$. Then, $g'_2 \circ_{\mathrm{Ad}}^{\mathrm{DR}} (g'_1 \circ_{\mathrm{Ad}}^{\mathrm{DR}} f')$ is in $A\langle\langle e_Z\rangle\rangle^{\mathrm{lim}}$ and we have

$$\lim\big(g'_2 \circ_{\mathrm{Ad}}^{\mathrm{DR}} (g'_1 \circ_{\mathrm{Ad}}^{\mathrm{DR}} f')\big) = \lim\big(g'_2 \circ_{\mathrm{Ad}}^{\mathrm{DR}} \lim(g'_1 \circ_{\mathrm{Ad}}^{\mathrm{DR}} f')\big)$$

Proof. The fact that $g_2' \circ_{\mathrm{Ad}}^{\mathrm{DR}} (g_1' \circ_{\mathrm{Ad}}^{\mathrm{DR}} f')$ is in $A\langle\langle e_Z\rangle\rangle^{\lim}$ has been already shown implicitly in the previous proof. Let us prove the rest of the statement. Take $(w_l)_{l\in\mathbb{N}}$ a sequence of the form $(e_0^l e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^r)_{l\in\mathbb{N}}$, with $i_1,\dots,i_{d+1} \in \{1,\dots,N\}$. The Proposition 2.3.5 applied two times gives a formula for $\big(g_2' \circ_{\mathrm{Ad}}^{\mathrm{DR}} (g_1' \circ_{\mathrm{Ad}}^{\mathrm{DR}} f')\big)[w_l]$. It is easy to check that it depends on l in the following way :

$$(3.3.5) \quad f(e_0, g_{z_1}, \dots, g_{z_N})[w_l] = \rho + \sum_{i=1}^{N} \sum_{u} \theta_{u,i} \sum_{b=1}^{l} (g_1')_{z_i}[e_0^{b-1}u] + \sum_{i',i''=1}^{N} \sum_{u',u''} \theta_{(u',u''),(i',i'')} \sum_{\substack{b,b'\geq 1\\ b+b'=l}} (g_1')_{z_i}[e_0^{b-1}u](g_2')_{z_{i'}}[e_0^{b'-1}u'']$$

where u, u', u'' run over certain connected subsequences of $e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^r$ and $\rho, \theta_{u,i}, \theta_{(u',u''),(i',i'')} \in A$ do not depend on l. (Indeed, the formula is a sum over quadruples $((sw_j)_{j\in J}, C, (s'w_{j'})_{j'\in J'}, C')$, where $(sw_j)_{j\in J}$ is a connected partition of an element $sw \in \mathcal{SW}(w)$, C is a coloring of $(sw_{j'})_{j'\in J'}$, $(uw_{j'})_{j'\in J'}$ is a connected partition of an element $s'w \in \mathcal{SW}\big(w/(((sw_j)_{j\in J}, C)\big)$, and C' is a coloring of $(s'w_{j'})_{j'\in J'}$. The set of these quadruples is partitioned into three subsets that give respectively the three terms of the formula : (in the sense of Definition 2.3.1) i) sw is not maximally at the left ; ii) sw is maximally at the left and uw is not ; iii) both sw and $s'w$ are maximally at the left.) The limit when $l \to \infty$ of the third term is $\sum_{i'=1}^{N} \sum_{i''=1}^{N} \sum_{u'} \theta_{(u',u''),(i',i'')} \sum_{b=1}^{+\infty} (g_1')_{z_i}[e_0^{b-1}u] \sum_{b'=1}^{+\infty} (g_2')_{z_{i'}}[e_0^{b'-1}u'']$. This last formula separates g_1' and g_2' in the factors depending on l ; it is then formal to show that the limit when $l \to \infty$ of equation (3.3.5) is a function of g_2' and $\lim \big(g_1' \circ_{\mathrm{Ad}}^{\mathrm{DR}} f\big)$, and then, that this function is exactly $\lim(g_2' \circ_{\mathrm{Ad}}^{\mathrm{DR}} \lim(g_1' \circ_{\mathrm{Ad}}^{\mathrm{DR}} f))$. □

In order to state some last algebraic properties, it is convenient to formulate differently the harmonic Ihara action. First, replacing $\tau(n)$ by $\tau(n_f)$ where n_f is a formal variable removes the necessity to restrict ourselves to summable elements of $\tilde{\Pi}(A)$ and one can define $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ through an action of $\tilde{\Pi}_{1,0}(A)$ on $A[[n_f]]\langle\langle e_Z\rangle\rangle^{\mathrm{const}}$. Let us consider the dual of this last map ; it can be seen as a map

$$(\circ_{\mathrm{har}}^{\mathrm{DR,RT}})^{\vee} : \mathcal{O}_{\mathrm{const}}^{ш,e_Z} \to \mathcal{O}_{\mathrm{const}}^{ш,e_Z} \otimes \tau(n_f)\widehat{\mathcal{O}^{ш,e_Z}}$$

where $\widehat{}$ is the completion of graded Hopf algebras and $\mathcal{O}_{\mathrm{const}}^{ш,e_Z}$ is the free $\mathbb{Q}$-vector space generated by words $e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^r$. We call $(\circ_{\mathrm{har}}^{\mathrm{DR,RT}})^{\vee}$ De Rham-rational harmonic Goncharov coaction. It can be factorized in a natural way as :

$$\mathcal{O}_{\mathrm{const}}^{ш,e_Z} \overset{(\circ_{\mathrm{har}}^{\mathrm{DR,RT}})^{\vee,T}}{\longrightarrow} \mathcal{O}_{\mathrm{const}}^{ш,e_Z} \otimes T\big(\tau(n_f)\widehat{\mathcal{O}^{ш,e_Z}} \otimes (\oplus_{i=1}^{N} \mathbb{Q}z_i)\big) \overset{id\otimes ev}{\longrightarrow} \mathcal{O}_{\mathrm{const}}^{ш,e_Z} \otimes \tau(n_f)\widehat{\mathcal{O}^{ш,e_Z}}$$

where T denotes the tensor algebra, and ev is defined through the dual of $g \mapsto (z \mapsto z_i z)_*(g^{-1}e_1 g)$ at $\tau(n_f)\widehat{\mathcal{O}^{ш,e_Z}} \otimes \mathbb{Q}z_i$, and summation over i. Because of the application to multiple harmonic sums, we will denote $e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^r \in \mathcal{O}_{\mathrm{const}}^{ш,e_Z}$ by $\Big(\begin{array}{c} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1; r \end{array}\Big)$, by $\Big(\begin{array}{c} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{array}\Big) = \Big(\begin{array}{c} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1; 0 \end{array}\Big)$, and we will say that such words of depth d and weight $s_d + \ldots + s_1 + r$; we denote by $\mathcal{O}_{\mathrm{const}*,d}^{ш,e_Z}$, the subspace generated by words of depth d.

Lemma 3.3.10. i) For any $d \in \mathbb{N}$, the map $(\circ_{\text{har}}^{\text{DR,RT}})^{\vee}$ sends

$$\mathcal{O}^{ш,e_Z,\text{const}}_{*,d} \to \oplus_{d'=0}^{d} \mathcal{O}^{ш,e_Z}_{\text{const}*,d'} \otimes T\big(\tau(n_f)\widehat{\mathcal{O}^{ш,e_Z}}_{*,d-d'} \otimes (\oplus_{i=1}^{N}\mathbb{Q}z_i)\big)$$

ii) The map $(\circ_{\text{har}}^{\text{DR,RT}})^{\vee}$ in depth 0 is $\big(\begin{array}{c} z_i \\ \emptyset; r \end{array} \big) \mapsto \big(\begin{array}{c} z_i \\ \emptyset; r \end{array} \big) \otimes 1$ for all $i \in \{1,\ldots,N\}$.

iii) The projection of any $(\circ_{\text{har}}^{\text{DR,RT}})^{\vee,T}\big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1; r \end{array} \big)$ onto $\mathcal{O}^{ш,e_Z}_{\text{const}*,0} \otimes T\big(\tau(n_f)\widehat{\mathcal{O}^{ш,e_Z}}_{*,d} \otimes (\oplus_{i=1}^{N}\mathbb{Q}z_i)\big)$ is $\displaystyle\sum_{a=0}^{r}\sum_{i=1}^{N} \big(\begin{array}{c} z_i \\ \emptyset; r-a \end{array} \big) \otimes \big(\sum_{b=0}^{+\infty} n_f^{b+s_d+\ldots+s_1} e_0^b e_{z_{i_{d+1}}} \ldots e_0^{s_1-1} e_{z_{i_1}} e_0^a \otimes z_i \big)$

Proof. i) follows from the analogous result for the Ihara action (Corollary 2.3.7) and the definition of $\circ_{\text{har}}^{\text{DR,RT}}$ (Proposition-Definition 3.3.5). ii) and iii) follow from the definition of $\circ_{\text{har}}^{\text{DR,RT}}$. □

3.3.4. *The De Rham-rational harmonic Frobenius map at $(\vec{1}_1, \vec{1}_0)$.*

Definition 3.3.11. We call the De Rham-rational harmonic α-th power Frobenius of $\pi_1^{\text{un}}(X_K)$ at $(\vec{1}_1, \vec{1}_0)$ the map

$$(\phi^{\alpha})_{\text{har}}^{\text{DR,RT}} : \begin{array}{c} \text{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\text{const}}) \to \text{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle^{\text{const}}) \\ f \mapsto \Phi_{p,\alpha} \circ_{\text{har}}^{\text{DR,RT}} f \end{array}$$

The Proposition 3.3.7 implies that the De Rham-rational harmonic Frobenius map is continuous for the product indexed by $\mathbb{N}$ of the $\mathcal{N}_D$-topology.

Let us remark that in all these proofs, the summability condition defiing $K\langle\langle e_Z\rangle\rangle_{\Sigma}$ can be weakened ; we used only that for $f \in \tilde{\Pi}_{z,0}(K)_{\Sigma}$, the series of the form $\sum_{l\in\mathbb{N}} f[e_0^l w]$ converge. The $\mathcal{N}_D$-topology can also be replaced by the topology of uniform convergence on the subsets $\{e_0^l w,\ l \in \mathbb{N}\}$ of $\mathcal{W}(e_Z)$.

3.4. **Limit of the other terms of equation (3.1.1) when $l \to \infty$ and $l \to l_{\lim} \in \mathbb{Z}_p$.** Let us consider again, as in §3.1, l_f a formal variable.

Definition 3.4.1. i) We define an element har_n of $K[l_f]\langle\langle e_Z\rangle\rangle^{\text{const}}$ by, for all indices :

$$\text{har}_n[e_0^l e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}} e_0^r] = \text{har}_n \big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1; r \end{array} \big)(l_f)$$

ii) We define $\text{har}_n^{(p^\alpha)} \in K[l_f]\langle\langle e_Z\rangle\rangle^{\text{const}}$ by $\text{har}_n^{(p^\alpha)}[w] = \text{har}_n[w^{(p^\alpha)}]$ for all words w.

When $r = 0$, this is a polynomial of degree 0 with respect to l_f. The limits of the terms of Proposition 3.2.6 that have not been treated in §3.2 are expressed by the following formulas.

Lemma 3.4.2. We fix $l_{\lim} \in \mathbb{Z}_p$.
i) We have

$$\lim_{\substack{|l|_{\mathbb{C}}\to\infty \\ |l-l_{\lim}|_p\to 0}} \tau(n)\, \text{Li}_{p,X_K^{\text{KZ}}}(z)[w_{l,r}] = \text{har}_n[e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}} e_0^r](l_f = l_{\lim})$$

ii) We have

$$\Big(\lim_{\substack{|l|_{\mathbb{C}}\to\infty \\ |l-l_{\lim}|_p\to 0}} \tau(n)\, \text{Li}^{\text{KZ}}_{p,X_K^{(p^\alpha)}}(z)\big(e_0, {\Phi_{p,\alpha}^{(z_1)}}^{-1} e_{z_1} \Phi_{p,\alpha}^{(z_1)}, \ldots, {\Phi_{p,\alpha}^{(z_N)}}^{-1} e_{z_N} \Phi_{p,\alpha}^{(z_N)}\big)[w_{l,r}][z^n \log(z)^0]\Big)_{n\in\mathbb{N}}$$

$$= \big(\Phi_{p,\alpha} \circ^{\mathrm{DR,RT}}_{\mathrm{har}} \mathrm{har}^{(p^\alpha)}\big)\left(\begin{array}{c} z_{i_{d+1}}^{p^\alpha}, \dots, z_{i_1}^{p^\alpha} \\ s_d, \dots, s_1 \end{array}\right)(l_f = l_{\lim})$$

ii') The term $n=1$ in ii) is

$$\sum_{b=0}^{+\infty}\sum_{a=0}^{r}\sum_{i=1}^{N} -z_i^{-p^\alpha} \binom{-(l_{\lim}-b)}{r-a} \big(\Phi_{p,\alpha}^{(z_i)^{-1}} e_{z_i} \Phi_{p,\alpha}^{(z_i)}\big)[w_{b,a}]$$

Proof. i) follows from Lemma 3.1.2.
ii) The term $\tau(n)\,\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}(z)\big(e_0, \Phi_{p,\alpha}^{(z_1)^{-1}} e_{z_1} \Phi_{p,\alpha}^{(z_1)}, \dots, \Phi_{p,\alpha}^{(z_N)^{-1}} e_{z_N} \Phi_{p,\alpha}^{(z_N)}\big)[w_{l,r}][z^n \log(z)^0]$ depends on l in the following way : $\sum_{b=0}^{l}\sum_{a=0}^{r} \binom{-(l-b)}{r-a} \times \nu_{b,a}$, where the factors $\nu_{b,a} \in K$ are independent of l and define a bounded function of b. When $|l|_{\mathbb{C}} \to \infty$ and $|l - l_{\lim}|_p \to 0$, one checks that the sequence $\binom{-(l-b)}{r-a}$ converges to $\binom{-(l_{\lim}-b)}{r-a}$ uniformly with respect to $b \in \mathbb{N}$. Then, $\displaystyle\sum_{b=0}^{l}\sum_{a=0}^{r}\binom{-(l-b)}{r-a} \times \nu_{b,a} \to \sum_{b=0}^{+\infty}\sum_{a=0}^{r}\binom{-(l_{\lim}-b)}{r-a} \times \nu_{b,a}$, and the result follows from the definition of the De Rham-rational harmonic Ihara action.
ii') $\mathrm{har}_1^{(p^\alpha)} = \mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}(z^{p^\alpha})[z^{p^\alpha}\log(z)^0] = \mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(p^\alpha)}}(z)[z\log(z)^0]$ is described as follows : it is given in depth one by $\mathrm{Li}_{p,X_K^{(p^\alpha)}}\mathrm{KZ}(z)[z][e_0^{l-b}e_{z_i}e_0^{r-a}] = \binom{-(l-b)}{r-a} z_i^{-p^\alpha}$, and, since all multiple harmonic sums of upper bound 1 are empty, it is equal to zero in all depths ≥ 2. Thus the result follows Lemma 3.3.10 iii). $\square$

3.5. **End of the proof of the De Rham part of Theorem I-2.a.**

Definition 3.5.1. Let $\mathrm{har}_{p^\alpha\mathbb{N}}, \mathrm{har}_{\mathbb{N}}^{(p^\alpha)} \in \mathrm{Map}(\mathbb{N}, K[l_f]\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$ be respectively the maps $n \mapsto \mathrm{har}_{p^\alpha n}$, $n \mapsto \mathrm{har}_n^{(p^\alpha)}$, with $\mathrm{har}_{p^\alpha n}, \mathrm{har}_n^{(p^\alpha)}$ are in the sense of Definition 3.4.1.

Since the Lemma 3.4.2 is true for all $l_{\lim} \in \mathbb{Z}_p$ and since the terms of its formulas depend polynomially on $l_{\lim}$, it lifts to equalities in $K[l_f]$. Thus, with the last definition above, the combination of Lemma 3.4.2 i) and ii), Proposition 3.1.6, and Corollary 3.2.2 implies the i) of Theorem I-2.a. When restricting the Theorem I-2.a to $n=1$, we obtain, by ii') of Lemma 3.4.2, the following formula, that is more general than Corollary I-2.a that corresponds to $r=0$:

Corollary 3.5.2. For all indices $\left(\begin{array}{c} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1; r \end{array}\right)$, we have the equality in $K[l_f]$:

$$\sum_{\substack{r_{d+1},\dots,r_1 \geq 0 \\ r_{d+1}+\dots+r_1 = r}} \binom{-l_f}{r_{d+1}} \prod_{i=1}^{d} \binom{-s_i}{r_i} \mathrm{har}_{p^\alpha}\left(\begin{array}{c} z_{i_{d+1}}, \dots, z_1 \\ s_d + r_d, \dots, s_1 + r_1 \end{array}\right) =$$

$$\sum_{b=0}^{+\infty}\sum_{a=0}^{r}\sum_{i=1}^{N} -z_i^{-p^\alpha}\binom{-(l_f-b)}{r-a}\big(\Phi_{p,\alpha}^{(z_i)^{-1}} e_{z_i} \Phi_{p,\alpha}^{(z_i)}\big)[e_0^b e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^a]$$

We may want to choose a linear basis of $K[l_f]$ over K and rephrase this last formula as a collection of equalities in K ; the most natural choice seems to be the basis given by the polynomials $\binom{-l_f}{r'}$, $r' \in \mathbb{N}$. The decomposition of the polynomials $\binom{-(l_f-b)}{r-a}$ in this basis is given by the equality $\frac{1}{(1-T)^{l_f-b}} = \frac{1}{(1-T)^{l_f}}(1-T)^b$, i.e. $\binom{-(l_f-b)}{r-a} = \sum_{r''=0}^{r-a}\binom{-l_f}{r''}\binom{b}{r-a-r''} = \sum_{r'=a}^{r}\binom{-l_f}{r-r'}\binom{b}{r'-a}$. Thus, the Corollary 3.5.2 can be reformulated by saying that, for all indices $\left(\begin{array}{c} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{array}\right)$, and $r' \in \mathbb{N}$ we have the following equality in K :

$$(3.5.1)\quad \sum_{\substack{r_d,\ldots,r_1\geq 0\\ r_d+\ldots+r_1=r'}} \prod_{i=1}^{d} \binom{-s_i}{r_i} \operatorname{har}_{p^\alpha} \left(\begin{array}{c} z_{i_{d+1}},\ldots,z_1 \\ s_d+r_d,\ldots,s_1+r_1 \end{array} \right)$$
$$= \sum_{b=0}^{+\infty}\sum_{a=0}^{r'}\sum_{i=1}^{N} -z_i^{-p^\alpha} \binom{b}{r'-a} \big(\Phi_{p,\alpha}^{(z_i)}{}^{-1} e_{z_i} \Phi_{p,\alpha}^{(z_i)}\big)[e_0^b e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}} e_0^a]$$

A study the sums $\sum_{\substack{r_d,\ldots,r_1\geq 0\\ r_d+\ldots+r_1=r'}} \prod_{i=1}^{d} \binom{-s_i}{r_i} \operatorname{har}_{p^\alpha} \left(\begin{array}{c} z_{i_{d+1}},\ldots,z_1 \\ s_d+r_d,\ldots,s_1+r_1 \end{array} \right)$ will be made in part III. We now prove the De Rham part of iii) of Theorem I-2.a.

Definition 3.5.3. i) Let $\mathcal{E}_{\mathrm{har}}^{\mathrm{DR,RT}} \subset \mathrm{Map}(\mathbb{N}, A\langle\langle e_Z\rangle\rangle^{\mathrm{const}})$ be the subset of elements $h=(h_n)_{n\in\mathbb{N}}$ such that the maps $n\in\mathbb{N}^* \mapsto h_n \left(\begin{array}{c} z_i \\ \emptyset, r \end{array} \right)$, $i\in\{1,\ldots,N\}$, $r\in\mathbb{N}$, are linearly independent over the ring $\mathfrak{A}(\mathbb{Z}_p)$ of rigid analytic functions of $(n\in)\mathbb{Z}_p$.
ii) Let $\mathcal{T}_{\mathrm{har}}^{\mathrm{DR,RT}}$ be the orbit of $\operatorname{har}_{\mathbb{N}}^{(p^\alpha)}$ for $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$.

Proposition 3.5.4. i) we have :
a') $\operatorname{har}_{\mathbb{N}}^{(p^\alpha)} \in \mathcal{E}_{\mathrm{har}}^{\mathrm{DR,RT}}$
b') $\mathcal{E}_{\mathrm{har}}^{\mathrm{DR,RT}}$ is stable by $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$
c') $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ restricted to $\mathcal{E}_{\mathrm{har}}^{\mathrm{DR,RT}}$ is free.
ii) In particular, $\mathcal{T}_{\mathrm{har}}^{\mathrm{DR,RT}} \subset \mathcal{E}_{\mathrm{har}}^{\mathrm{DR,RT}}$ and $\mathcal{T}_{\mathrm{har}}^{\mathrm{DR,RT}}$ is a torsor for the De Rham-rational harmonic Ihara action containing $\operatorname{har}_{\mathbb{N}}^{(p^\alpha)}$.

Proof. i) a') For all $i\in\{1,\ldots,N\}$, $r\in\mathbb{N}$, we have $\operatorname{har}_n \left(\begin{array}{c} z_i \\ \emptyset; r \end{array} \right) = \binom{-l_f}{r} z_i^{-n} \in K[l_f]$; since the polynomials $\binom{-l_f}{r}$ are linearly independent over any ring of characteristic 0, in particular $\mathrm{Map}(\mathbb{N},K)$, we are reduced to prove the desired linear independence property for $r=0$; that follows easily by considering n restricted to the classes of congruences modulo N. b') This follows from part ii) of Lemma 3.3.10. c') One proves easily by induction on d that the harmonic Ihara action truncated to depths at most d is free by iii) of Lemma 3.3.10.
ii) is an immediate consequence of i). ☐

4. A "rational setting" for multiple harmonic sums

In §4 and §5, our goal is to develop a theory of a kind of Frobenius on multiple harmonic sums in a way that makes sense independently from the De Rham setting. We will nevertheless make a few references to the De Rham setting by stating some immediate interpretations or heuristic analogies, in order to clarify the exposition, and slightly anticipating on the De Rham-rational comparison of §6. In §4.1 we define a notion of multiple harmonic sums that is more general than the one that we used until now ; in §4.2, we state some first rules of computations on them ; in §4.3 we define a counterpart of the motivic Galois action of $\mathbb{G}_m$; in §4.4 we partially define a counterpart of the composition of series in $K\langle\langle e_Z\rangle\rangle$, i.e. essentially of the adjoint Ihara action.

4.1. **A class of elementary p-adic functions containing multiple harmonic sums.** The purpose of the next definitions is the maximality of the results of §5, but a certain part of this degree of generality is nevertheless required for our proofs even if we intend to restrict to the usual multiple harmonic sums.

Definition 4.1.1. Let G_α be the subgroup of $\mathrm{Hom}_{\mathrm{gp}}(K^*, K^*)$ made of elements χ such that, for all $\epsilon \in K$ satisfying $|\epsilon|_p \leq \frac{1}{p^\alpha}$, we have an absolutely convergent expansion $\chi(1+\epsilon) = \sum_{l\geq 0} \chi^{((l))}(1)\epsilon^l$ with $(\chi^{((l))}(1))_{l\in\mathbb{N}} \in K^{\mathbb{N}}$.

In particular, the elements of G_α are locally analytic maps ; moreover, G_α contains the elements $n_j \mapsto \frac{1}{n_j^{s_i}}$, $s_j \in \mathbb{N}^*$. We fix $d \in \mathbb{N}^*$.

Definition 4.1.2. i) Let $\mathcal{W}^{'}(e_Z)$ be the set of sequences $\left(\begin{array}{c} z_{i_{d+1}}, \ldots\ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right)$ with $d \in \mathbb{N}^*$, where $i_1, \ldots, i_{d+1} \in \{1, \ldots, N\}$, and $\chi_1, \ldots, \chi_d \in \mathrm{Vect}_K(G_\alpha)$; d is called the depth of such a sequence.
ii) For all $d \in \mathbb{N}^*$, let $\mathcal{W}^{'}_{*,d}(e_Z) \subset \mathcal{W}^{'}_{*}(e_Z)$ be the subset of words of depth d.

Definition 4.1.3. For any $w = \left(\begin{array}{c} z_{i_{d+1}}, \ldots\ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right)$ in $\mathcal{W}^{'}_{*,d}(e_Z)$, and $(n_0, n) \in \mathbb{Z}^2$ such that $0 \leq n_0 < n$ or $n_0 < n \leq 0$, we call multiple harmonic sum the number

$$\sigma_{n_0,n}(w) = z_{i_1}^{n_0} \Bigg(\sum_{n_0 < n_1 < \ldots < n_d < n} \prod_{j=1}^{d} \left(\frac{z_{i_{j+1}}}{z_{i_j}} \right)^{n_j} \chi_j(n_j) \Bigg) \frac{1}{z_{i_{d+1}}^n}$$

Definition 4.1.4. i) Let $\mathrm{wt} : (G_\alpha, \times) \to (K, +)$, the weight map, be the morphism of groups that sends $\chi \mapsto -\frac{\log_p(\chi(p))}{\log_p(p)}$.

ii) For any $w = \left(\begin{array}{c} z_{i_{d+1}}, \ldots\ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right)$ in $\mathcal{W}^{'}_{*,d}(e_Z)$, and $(n_0, n) \in \mathbb{Z}^2$ such that $0 \leq n_0 < n$ or $n_0 < n \leq 0$, we call weighted multiple harmonic sum the number :

$$\mathrm{har}_{n_0,n}(w) = (n - n_0)^{\mathrm{wt}(\chi_d \cdots \chi_1)}\ \sigma_{n_0,n}(w)$$

The definition extends to $\chi_d, \ldots, \chi_1 \in \mathrm{Vect}_K(G_\alpha) = K[G_\alpha]$ by multilinearity.

Notation 4.1.5. i) When the definition makes sense (this is not always the case for $n_0 = 0$), we denote by $\tilde{\sigma}_{n_0,n}$, resp. $\widetilde{\mathrm{har}}_{n_0,n}$, the function defined like $\sigma_{n_0,n}$ except that the sum goes over $n_0 \leq n_1 < \ldots < n_d < n$, i.e. includes the $n_1 = n_0$ terms.
ii) For all $n \in \mathbb{N}^*$, we denote by $\sigma_n = \sigma_{0,n}$, $\tilde{\sigma}_n = \tilde{\sigma}_{0,n}$, $\mathrm{har}_n = \mathrm{har}_{0,n}$, $\widetilde{\mathrm{har}}_n = \widetilde{\mathrm{har}}_{0,n}$.
iii) When $z_{i_{d+1}} = \ldots = z_{i_1} = 1$, we denote as in the De Rham setting by $\left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = (\chi_d, \ldots, \chi_1)$.
iv) When for all $i \in \{1, \ldots, d\}$, χ_i is a map of the form $n_i \mapsto n_i^{-s_i}$, with $s_i \in \mathbb{Z}$ (not necessarily $s_i \in \mathbb{N}^*$), we denote by $\left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \right)$. With this notation, the notations har_n, σ_n, $\tilde{\sigma}_n$ are compatible with the one that we use in the De Rham setting.

As in the De Rham setting, we call prime (weighted) multiple harmonic sums the (weighted) multiple harmonic sums $\mathrm{har}_n, \widetilde{\mathrm{har}}_n, \sigma_n, \tilde{\sigma}_n$ such that n is a power of p.

Remark 4.1.6. We have, for all $\chi \in G_\alpha$, and for all $a \in \mathbb{N}^*$, $(p^a)^{\mathrm{wt}(\chi)} = \frac{1}{\chi(p^a)}$; so, when $n - n_0$ is a power of p, we have a slight simplification of the formula defining weighted multiple harmonic sums :

$$\mathrm{har}_{n_0,n} \left(\begin{array}{c} z_{i_{d+1}}, \ldots\ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = z_{i_1}^{n_0} \Bigg(\sum_{n_0 < n_1 < \ldots < n_d < n} \prod_{j=1}^{d} \left(\frac{z_{i_{j+1}}}{z_{i_j}} \right)^{n_j} \chi_j \left(\frac{n_j}{n - n_0} \right) \Bigg) \frac{1}{z_{i_{d+1}}^n}$$

Remark 4.1.7. Certain of the computations in the next paragraphs remain true if one replaces the factor $\frac{z_{i_1}^{n_0}}{z_{i_{d+1}}^{n}}$ of multiple harmonic sums by any $\frac{z_{i'_1}^{n_0}}{z_{i'_{d+1}}^{n}}$, $i'_1, i'_{d+1} \in \{1, \ldots, N\}$, such that $z_{i'_1} z_{i'_{d+1}}^{-1} = z_{i_1} z_{i_{d+1}}^{-1}$.

4.2. **Formulas of change of variables and composition of paths.** We propose an analogy that we will explain more precisely in II-3 : the multiple harmonic sums of §4.1 are kinds of "iterated integrals on the space $\mathbb{Z}$", where the domain of iterated summation $n_0 < n_1 < \ldots < n_d < n$ is a kind of "path from n_0 to n in $\mathbb{Z}$". Under this analogy, the following statements are analogues of standard rules of computations for iterated integrals. We write the formulas in the weighted case ; they are of course also true in the non-weighted case. They will be used in §5.

4.2.1. *Reduction to $n_0 \geq 0$.* We start with the simplest formula of change of variables :

Lemma 4.2.1. Let $n_0, n \in \mathbb{Z}$ such that $n_0 < n < 0$. We have, for all indices

$$\mathrm{har}_{n_0,n} \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = \left(\prod_{i=1}^{d} \chi_i(-1) \right) \mathrm{har}_{-n,-n_0} \left(\begin{array}{c} z_{i_1}, \ldots, z_{i_{d+1}} \\ \chi_1, \ldots, \chi_d \end{array} \right)$$

We are thus reduced to study of multiple harmonic sums attached to (n_0, n) such that $0 \leq n_0 < n$.

4.2.2. *Reduction to $n_0 = 0$ by splitting.* The next formula is an analogue of the formula of composition of paths of iterated integrals.

Lemma 4.2.2. For all $n, n_0 \in \mathbb{N}^*$, such that $n_0 < n$, we have :

$$\mathrm{har}_n \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = \sum_{d'=0}^{d} \widetilde{\mathrm{har}}_{n_0,n} \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_{d'+1}} \\ \chi_d, \ldots, \chi_{d'+1} \end{array} \right) \mathrm{har}_{n_0} \left(\begin{array}{c} z_{i_{d'+1}}, \ldots, z_{i_1} \\ \chi_{d'}, \ldots, \chi_1 \end{array} \right)$$

By this lemma and the following relation between $\mathrm{har}_{n_0,n}$ and $\widetilde{\mathrm{har}}_{n_0,n}$ for $n_0 > 0$:

$$\widetilde{\mathrm{har}}_{n_0,n} \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = (n - n_0)^{\mathrm{wt}(\chi_1)} \chi_1(n_0) \, \mathrm{har}_{n_0,n} \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_2} \\ \chi_d, \ldots, \chi_2 \end{array} \right)$$

we obtain a formula for $\mathrm{har}_{n_0,n}$ in terms of $\mathrm{har}_n = \mathrm{har}_{0,n}$.

4.2.3. *Reduction to $n_0 = 0$ by shifting.* We now write a second formula of change of variables. Let $n \in \mathbb{N}$ and $\delta \in \mathbb{N}$; we have by definition

$$\mathrm{har}_{\delta,\delta+n} \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = \delta^{\mathrm{wt}(\chi_d, \ldots, \chi_1)} \frac{z_{i_1}^{\delta}}{z_{i_{d+1}}^{\delta+n}} \sum_{0<n_1<\ldots<n_d<n} \prod_{j=1}^{d} \left(\frac{z_{i_{j+1}}}{z_{i_j}} \right)^{n_j} \chi_j(n_j + \delta)$$

and as a consequence :

Lemma 4.2.3. Assume moreover that, for all $n' \in [1, n-1]$, we have $v_p(n') < v_p(\delta)$; for example, $n = p^{\alpha}$, and $\delta = up^{\alpha}$, $u \in \mathbb{N}$. Then, we have :

$$\mathrm{har}_{\delta,\delta+n} \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d, \ldots, \chi_1 \end{array} \right) = \sum_{l_1, \ldots, l_d \geq 0} \left(\prod_{j=1}^{d} \delta^{l_j} \chi_j^{((l_i))}(1) \right) \mathrm{har}_n \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_d \psi_{-l_d}, \ldots, \chi_1 \psi_{-l_1} \end{array} \right)$$

Proof. One rewrites each factor $\chi_j(n_j + \delta) = \chi_j(n_j)\chi_j(1 + \frac{\delta}{n_j})$ according to the series expansion of χ_j at 1. $\square$

For usual prime weighted multiple harmonic sums, this implies :

(4.2.1)

$$\mathrm{har}_{up^\alpha,(u+1)p^\alpha}\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array} \Big) = \sum_{(l_d,\ldots,l_1)\in\mathbb{N}^d} \bigg(\prod_{i=1}^{d} \binom{-s_i}{l_i} (up^\alpha)^{l_i}\bigg) \mathrm{har}_{p^\alpha}\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d+l_d,\ldots,s_1+l_1 \end{array} \Big)$$

Aside from the application of this formula in §5, this equation will have applications later where we will study intrinsically the shifted prime multiple harmonic sums.

4.3. **Rational analogue of the weight grading.** We recall from I-1 the following definition : let $\mathcal{P}_N$ be the set of prime numbers that are prime to N, and let

$$\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}} = \mathrm{Vect}_{\mathbb{Q}}\big\{\big(\sum_{L\in\mathbb{N}} F_L(\xi,\ldots,\xi^{N-1})\prod_{\eta=1}^{\eta_0}\mathrm{har}_{p^\alpha}(w_{L,\eta})\big)_{(p,\alpha)} \mid (*)\big\} \subset \prod_{\substack{(p,\alpha)\\ \in\mathcal{P}_N\times\mathbb{N}^*}} \mathbb{Q}_p(\mu_N)$$

where $(*)$ means that $(w_{L,1})_{L\in\mathbb{N}},\ldots,(w_{L,\eta_0})_{L\in\mathbb{N}}$ ($\eta_0\in\mathbb{N}^*$) are sequences of words satisfying $\sum_{\eta=1}^{\eta_0}\mathrm{weight}(w_{L,\eta}) \to_{l\to\infty} \infty$ and $\limsup_{L\to\infty}\sum_{\eta=1}^{\eta_0}\mathrm{depth}(w_{L,\eta}) < \infty$, and $(F_L)_{L\in\mathbb{N}}$ is a sequence of elements of $\mathbb{Q}$ if $N=1$, resp. $\mathbb{Q}[T_1,\ldots,T_{N-1},\frac{1}{T_1},\ldots,\frac{1}{T_{N-1}},\frac{1}{T_1-1},\ldots,\frac{1}{T_{N-1}-1}]$ if $N\neq 1$. We showed in I-1, and we also reproved in I-2, that the sequences $(\zeta_{p,\alpha})_{(p,\alpha)\in\mathcal{P}_N\times\mathbb{N}^*}$ are in $\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}$. We will encounter the following space.

Definition 4.3.1. Let n_f be a formal variable, and let $\mathfrak{A}(\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}) = \{\sum_{L\geq 0}\kappa_L n_f^L \mid \kappa_L \in \widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}, |\kappa_L|_p \to_{L\to+\infty} 0\}$

The following will apply to certain subspaces of $\mathfrak{A}(\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}})$ defined through restrictions concerning the weight and depth of prime weighted multiple harmonic sums, similar to the ones defined in I-1, §1 for $\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}$, and through the formulas for $\zeta_{p,\alpha}$.

Definition 4.3.2. Let :

$$\widehat{\tau}^{\mathrm{RT}} : \begin{array}{c} \mathcal{O}_K\times\mathfrak{A}(\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}) \to \mathfrak{A}(\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}) \\ (\lambda,\sum_{L\in\mathbb{N}}\kappa_L n_f^L) \mapsto \sum_{L\in\mathbb{N}}\kappa_L(\lambda n_f)^L \end{array}$$

If we want to deal with all indices introduced in §4.1 we can replace $\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}$ by the larger algebra where the words w_{L,η_i} are replaced by those more general words w_{L,η_i} in the sense of §4.1 such that we have $v_p(\mathrm{har}_{p^\alpha}(w_{L,\eta_i})) \geq \mathrm{wt}(w_{L,\eta_i})$.

4.4. **Rational analogue of the setting for composition of series in $K\langle\langle e_Z\rangle\rangle$.** As we saw in §2.3.1, the composition of series in $K\langle\langle e_Z\rangle\rangle$, that is to say essentially the adjoint Ihara action, is expressed by the operations such as taking subwords and taking quotient words of words on e_Z. We define a partial analogue of this setting, that we will use in §5. We fix $d\in\mathbb{N}^*$ and we denote by $\mathfrak{P}(\{1,\ldots,d+1\})$ the set of parts of $\{1,\ldots,d+1\}$.

4.4.1. *Setting for subwords.*

Definition 4.4.1. i) We call the set of words of relative depth d and denote by $\mathcal{W}_{*,d}^{\mathrm{rel}}$ the set of couples (w,S) where $w=\Big(\begin{array}{c} z_{i_{d'+1}},\ldots,z_{i_1} \\ \chi_{d'},\ldots,\chi_1 \end{array}\Big)$ is a word of depth $d'\in\{1,\ldots,d\}$ and S is an element of $\mathfrak{P}(\{1,\ldots,d+1\})$ of cardinal $d'+1$.
ii) We call relative (weighted) multiple harmonic sums of depth d a couple $(\sigma_n(w),S)$, resp. $(\mathrm{har}_n(w),S)$, $n\in\mathbb{N}$, where $(w,S)\in\mathcal{W}_{*,d}^{\mathrm{rel}}$.

Concretely, we will write relative multiple harmonic sums as usual multiple harmonic sums except that the indices $i_1,\ldots,i_{d'+1}$ of the corresponding word $w = \begin{pmatrix} z_{i_{d'+1}},\ldots,z_{i_1} \\ \chi_{d'},\ldots,\chi_1 \end{pmatrix}$ will be denoted by $i_{I_1},\ldots,i_{I_{d'+1}}$ where $S = \{I_1 < \ldots < I_{d'+1}\} \in \mathcal{P}(\{1,\ldots,d+1\})$. In other terms, $\mathcal{W}^{\mathrm{rel}}_{*,d}$ is the image of a natural map of restriction :

$$\mathrm{restr} : \mathcal{W}_{*,d}(e_Z) \times \mathcal{P}(\{1,\ldots,d+1\}) \to \mathcal{W}^{\mathrm{rel}}_{*,d}(e_z)$$

4.4.2. *Setting for quotient words.* We fix $S \in \mathfrak{P}(\{1,\ldots,d+1\})$.

Definition 4.4.2. We call the boundary of S and denote by ∂S the subset of S made of the elements x such that $x-1 \notin S$ or $x+1 \notin S$.

Definition 4.4.3. Let the $\mathbb{N}$-topology on S be the one generated by the segments $[a,b] \subset S$ such that $(a,b) \in (\partial S)^2$.

One checks that S is the disjoint union of these segments, that they are both open and closed in S, and that they are the connected components of S. In particular, S is connected if and only if it is a segment $[a,b]$, and the connected subsets of S are the segments. The operation of taking quotient words, that will appear in §5, will be related to the following definitions.

Definition 4.4.4. A connected partition of S is a partition of S into connected subsets, i.e. into segments.

Definition 4.4.5. Let $\mathrm{dec}(S) = S - \partial S$.

5. Proofs in the rational setting

We study in §5.1 the multiple harmonic sums whose indices $\chi_d,\ldots,\chi_1$ are polynomials, as functions of the bound n. We deduce from it in §5.2 a family of algebraic relations between multiple harmonic sums that we call elimination of positive powers. In §5.3 we study the operation of multiplying the upper bound of weighted multiple harmonic sums by a given integer, in particular by p^α. In §5.4, combining §5.2 and §5.3 we obtain a first version of the rational harmonic Ihara action, and prove part of Theorem I-2.a ii). In §5.5, we lift the rational harmonic Ihara action into an operation that is a first version of the rational Ihara action on the bundle of paths starting at $\vec{1}_0$, and prove part of Theorem I-2.b ii). The objects from §5.4 and §5.5 will be reindexed in §6 by the comparison with the De Rham setting.

5.1. **Computation of multiple harmonic sums whose indices are polynomials.** Since the space $\mathrm{Vect}_K(G)$ of §4.1 contains polynomial functions, one can consider :

Definition 5.1.1. Let $\mathcal{W}^{\mathrm{pol}}(e_Z)$ be the set of sequences $w = \begin{pmatrix} z_{i_{d+1}},\ldots,z_{i_1} \\ \chi_d,\ldots,\chi_1 \end{pmatrix}$ such that $\chi_d,\ldots,\chi_1$ are polynomials $P_d,\ldots,P_1$.

We recall that $z_i = z_1^i = \xi^i$ for all $i \in \{1,\ldots,N\}$, and in particular $z_N = 1$. We will consider here formal variables $zf_1,\ldots,zf_N$ without imposing any relation between them, and formal variables T_{inf} and T_{sup} that encode the lower bound and upper bound of a multiple harmonic sum.

Proposition-Definition 5.1.2. There exists a unique map

$$\mathrm{pol} : \mathcal{W}^{\mathrm{pol}}(e_Z) \to \oplus_{i,j=1}^N (zf_i)^{T_{\mathrm{inf}}} (zf_j)^{T_{\mathrm{sup}}} \mathbb{Q}(\xi)[T_{\mathrm{inf}}, T_{\mathrm{sup}}]$$

that is multilinear with respect to $(P_d,\ldots,P_1)$ - we will denote it by

$$\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ P_d,\ldots,P_1 \end{array} \Big) \mapsto \sum_{i,j=1}^{N} \sum_{m,m'=0}^{M(P_d,\ldots,P_1)} \mathcal{B}_{(m,m'),(i,j)} \Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ P_d,\ldots,P_1 \end{array} \Big) T_{\sup}^{m} {T_{\inf}}^{m'} (zf_i)^{T_{\sup}} (zf_j)^{T_{\inf}}$$

with $M(P_d,\ldots,P_1)\in\mathbb{N}$ - and such that for all $n'<n\in\mathbb{N}$, we have

$$\sigma_{n',n}|_{\mathcal{W}^{\mathrm{pol}}(e_Z)} = \mathrm{pol}(z_1,\ldots,z_N,n',n)$$

Moreover, for all $\delta_1,\ldots,\delta_d\in\mathbb{N}^*$, if $\deg(P_i)\leq\delta_i$ for all i, then we can take $M(P_d,\ldots,P_1)\leq \delta_1+\ldots+\delta_d+d$; and finally, we have :

$$v_p\big(\mathcal{B}_{(m,m'),(i,j)}\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ P_d,\ldots,P_1 \end{array} \Big)\big) \geq -d-d\frac{\log(\delta_1+\ldots+\delta_d+d)}{\log(p)}$$

Proof. The uniqueness is clear. By the formula of splitting of multiple harmonic sums (Lemma 4.2.2), it is sufficient to treat the case where $n'=0$. By induction on d, it is sufficient to treat the case where $d=1$. By multilinearity of a multiple harmonic sums with respect to $(P_d,\ldots,P_1)$, it is sufficient to treat the case where $P_d,\ldots,P_1$ are monomials. Then we are reduced to a statement that is already proved in I-1 :
If $z=1$, we haven for all $n\in\mathbb{N}^*$, and $m\in\mathbb{N}$, $\sum_{u_1=1}^{u-1} u_1^m = \sum_{l=1}^{m+1}\mathcal{B}_l^m u^l$, with $\mathcal{B}_l^m = \frac{1}{m+1}\binom{m+1}{l}B_{l+1-m}$ where B denotes Bernoulli numbers, and the bound of valuation follows from Von Staudt-Clausen's theorem and the fact that $v_p(\frac{1}{l+1})\geq -\frac{\log(l+1)}{\log(p)}$.
If $z\neq 1$, we can write $\sum_{n_1=0}^{n-1} n_1^l T^{n_1} = \big(T\frac{d}{dT}\big)^l\big(\frac{1-T^n}{1-T}\big)$ where T is a formal variable ; the bound of valuation follows from that, for all $i\in\{1,\ldots,N-1\}$, we have $|z_i|_p=|z_i-1|_p=1$. $\square$

Remark 5.1.3. The multilinear forms $\mathcal{B}$ above depend on $z_1,\ldots,z_N$ when $N\neq 1$ through rational functions in $\mathbb{Z}[T_1,\ldots,T_{N-1},\frac{1}{T_1},\ldots,\frac{1}{T_{N-1}},\frac{1}{T_1-1},\ldots,\frac{1}{T_{N-1}-1}]$, applied to $(z_1,\ldots,z_{N-1})$.

Remark 5.1.4. The multiple harmonic sums σ and $\tilde{\sigma}$ whose indices are in $\mathcal{W}^{\mathrm{pol}}(e_Z)$ are of course equal to each other unless some of the l_i's are equal to 0 and are expressed in terms of each other as follows : if $l_1,\ldots,l_d\in\mathbb{N}$, let u such that $l_u\neq 0$ and $l_1=\ldots=l_{u-1}=0$, we have $\tilde{\sigma}_n\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ -l_d,\ldots,-l_1 \end{array} \Big) = \sum_{j=1}^{u}\sigma_n\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_j} \\ -l_d,\ldots,-l_j \end{array} \Big)$ and $\sigma_n\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ -l_d,\ldots,-l_u,0,\ldots,0 \end{array} \Big) = \sum_{j=0}^{u-1}(-1)^j\tilde{\sigma}_n\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_j} \\ -l_d,\ldots,-l_j \end{array} \Big)$.

5.2. **Elimination of the positive powers.** In the De Rham setting, one can consider iterated integrals not only of $\frac{dz}{z},\frac{dz}{z-\xi^1},\ldots,\frac{dz}{z-\xi^N}$, but also of other differential forms over X_K. However, by the description of $\mathcal{O}(\pi_1^{\mathrm{un,DR}}(X_K,\omega_{\mathrm{DR}}))$, the iterated integrals of these particular forms essentially generate all iterated integrals over X_K. The next result is a rational analogue of this fact.

5.2.1. *Setting.* Let $\chi_1 : n\mapsto n\in G_\alpha$; for all $t\in\mathbb{Z}$, the map $\chi_1^t : n\mapsto n^t$ is in G_α. We consider the multiple harmonic sums associated with the words of the following form.

Definition 5.2.1. Let $\mathcal{W}^{\mathrm{loc}}(e_Z)\subset\mathcal{W}'(e_Z)$ be the set of words of the form $\Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ \chi_1^{-t_d},\ldots,\chi_1^{-t_1} \end{array} \Big) = \Big(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ t_d,\ldots,t_1 \end{array} \Big)$ with $i_1,\ldots,i_{d+1}\in\{1,\ldots,N\}$, $t_d,\ldots,t_1\in\mathbb{Z}$.

We write "loc" as a reference to the localization of the non-commutative ring $\mathbb{Q}\langle e_Z\rangle$ (equipped with the concatenation product) at the multiplicative part generated by e_0. For our purposes, we must separate the parts associated with positive and negative powers t_i.

Notation 5.2.2. For $w = \left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ t_d,\ldots,t_1 \end{array}\right) \in \mathcal{W}^{\mathrm{loc}}(e_Z)$,
i) let $S^-(w), S^+(w)$ be the subsets of $\{1,\ldots,d\}$ such that we have $t_i<0$, resp. $t_i \geq 0$.
ii) Let $r \in \mathbb{N}$ be the number of connected components of $S^+(w)$ in the sense of §4.4.2, and we denote these connected components by $[I_1+1, J_1-1], \ldots, [I_r+1, J_r-1]$, with $I_1 < J_1 < I_2 < J_2 < \ldots < I_r < J_r$. We also write $J_0 = 0$ and $I_{r+1} = d+1$
iii) let us write $-t_i = s_i$ if $t_i<0$, and $-t_i = l_i$ if $t_i \geq 0$.

We have then the following expression, that can also be written in terms of relative words in the sense of §4.4.1 :

(5.2.1)
$$\sigma_n(w) = \sum_{\substack{0=n_{J_0}<n_{I_1}<n_{J_1} \\ <\ldots n_{J_r}<n_{I_{r+1}}=n}} \prod_{k=0}^{r} \sigma_{n_{J_k},n_{I_{k+1}}}\left(\begin{array}{c} z_{i_{I_k}},\ldots,z_{i_{J_k}} \\ s_{I_k},\ldots,s_{J_k} \end{array}\right) \prod_{k=1}^{r} \tilde{\sigma}_{n_{I_k},n_{J_k}}\left(\begin{array}{c} z_{i_{J_k}},\ldots,z_{i_{I_k+1}} \\ -l_{J_k-1},\ldots,-l_{I_k+1} \end{array}\right)$$

5.2.2. *Definition and recursive formula.* Combining the last equation and the main result of §5.1 we obtain the following :

Lemma 5.2.3. We have, for $w \in \mathcal{W}^{\mathrm{loc}}(e_Z)$ with the notations above :
$$\sigma_n(w) = \sum_{\substack{0=n_{J_0}<n_{I_1}<n_{J_1} \\ <\ldots n_{J_r}<n_{I_{r+1}}=n}} \prod_{k=0}^{r} \sigma_{n_{I_k},n_{J_k}}\left(\begin{array}{c} z_{i_{I_k}},\ldots,z_{i_{J_k}} \\ s_{I_k},\ldots,s_{J_k} \end{array}\right) \prod_{k=1}^{r} \mathrm{pol}(z_1,\ldots,z_N,n_{I_k},n_{J_k})\left(\begin{array}{c} z_{i_{J_k}},\ldots,z_{i_{I_k+1}} \\ -l_{J_k-1},\ldots,-l_{I_k+1} \end{array}\right)$$

This expression involves in general both positive and negative powers of n_{I_k} and n_{J_k}. Let us consider again the formal variables of §5.1. Separating positive and negative powers of T_{inf} and T_{sup}, the vector space :
$$\oplus_{i,j=1}^{N} (zf_i)^{T_{\mathrm{inf}}}(zf_j)^{T_{\mathrm{sup}}}\mathbb{Q}(\xi)[T_{\mathrm{inf}},T_{\mathrm{sup}},\frac{1}{T_{\mathrm{inf}}},\frac{1}{T_{\mathrm{sup}}}] \xrightarrow{\sim}$$
is the direct sum of
$$\left(\oplus_{i,j=1}^{N} (zf_i)^{T_{\mathrm{inf}}}(zf_j)^{T_{\mathrm{sup}}} \operatorname{Vect}_{\mathbb{Q}(\xi)}\{T_{\mathrm{inf}}^{s_{\mathrm{inf}}}T_{\mathrm{inf}}^{s_{\mathrm{sup}}} \mid s_{\mathrm{inf}} \geq 0 \text{ and } s_{\mathrm{sup}} \geq 0\}\right)$$
and the three other analogous subspaces defined by respectively $s_{\mathrm{inf}} \geq 0$ and $s_{\mathrm{sup}} < 0$, $s_{\mathrm{inf}} < 0$ and $s_{\mathrm{sup}} \geq 0$, $s_{\mathrm{inf}} < 0$ and $s_{\mathrm{sup}} < 0$.

Definition 5.2.4. For $(\epsilon_{sup},\epsilon_{inf}) \in \{-1,1\}^2$, we denote by $proj_{T_{sup}^{\epsilon_{sup}},T_{inf}^{\epsilon_{inf}}}$ the projection onto the corresponding subspace.

With the notations above, we have $\partial S^-(w) = \{I_1, J_1, \ldots, I_r, J_r, d+1\}$. We can define an analogue of these projection maps with a higher number of formal variables corresponding to the elements of $\partial S^-(w)$. We obtain the following object.

Proposition-Definition 5.2.5. Let the map of elimination of the positive powers :
$$\mathrm{el} : \mathcal{W}^{\mathrm{loc}}(e_Z) \longrightarrow \operatorname{Vect}_{\mathbb{Q}}(\mathcal{W}(e_Z)) \otimes \left(\oplus_{i,j=1}^{N} (zf_i)^{T_{\mathrm{inf}}}(zf_j)^{T_{\mathrm{sup}}}\mathbb{Q}(\xi)[T_{\mathrm{inf}},T_{\mathrm{sup}}]\right)$$

defined by induction on d by combining the previous formulas and separating the positive and negative powers of n_{I_k} and n_{J_k}. Then, we have :

$$\mathrm{har}_n = \mathrm{mult}_K \circ (\mathrm{har}_n \otimes \text{evaluation at } T_{\inf} = 0,\ T_{\sup} = n) \circ \mathrm{har}_n$$

The map el appears as a sum over $P \subset \partial S^{-}(w)$, and the inductivity concerns the P that are non-empty, for which the definition will be applied to set of indices of strictly lower relative depth.

Remark 5.2.6. We will prove in II.2 a property of uniqueness of the map el among a certain class of functions.

Let us give some examples in low depth.

Example 5.2.7. In depth one and for $\mathbb{P}^1 - \{0,1,\infty\}$, the map el is trivial ; in depth two and for $\mathbb{P}^1 - \{0,1,\infty\}$, we have two trivial formulas :

$$el(s_2, s_1) = (s_2, s_1) \otimes 1$$

$$el(-l_2, -l_1) = 1 \otimes \tilde{\sigma}_T(-l_2,-l_1) = 1 \otimes \textstyle\sum_{k=1}^{l_2+l_1+1} \mathcal{B}_k^{l_2,l_1} T^k$$

and two other formulas : if $l_2 + 1 \leq s_1 - 1$ then

$$el(-l_2, s_1) = (s_1) \otimes \textstyle\sum_{k_2=1}^{l_2+1} \mathcal{B}_{k_2}^{l_2} T^{k_2} - \sum_{k=1}^{l_2+1} (s_1 - k_2) \otimes \mathcal{B}_{k_2}^{l_2}$$

otherwise :

$$el(-l_2, s_1) = (s_1) \otimes \textstyle\sum_{k_2=1}^{l_2+1} \mathcal{B}_{k_2}^{l_2} T^{k_2} - \sum_{k=1}^{s_1-1} (s_1 - k_2) \otimes \mathcal{B}_{k_2}^{l_2}$$
$$-1 \otimes \textstyle\sum_{k_2=s_1}^{l_2+1} \sum_{k_1=0}^{k_2+1-s_1} \mathcal{B}_{k_2}^{l_2} \mathcal{B}_{k_1}^{k_2-s_1} T^{k_1}$$

5.2.3. *Non-recursive formula.* The explicit and non-recursive formula for the elimination of positive powers will be indexed by the following objects. We define them as functions of $w = \left(\begin{smallmatrix} z_{i_{d+1}}, \ldots, z_{i_1} \\ \chi_1^{t_d}, \ldots, \chi_1^{t_1} \end{smallmatrix}\right) \in \mathcal{W}^{loc}(e_Z)$, but they depend only on the sequence of signs of $t_d, \ldots, t_1$.

Definition 5.2.8. For $w \in \mathcal{W}^{\mathrm{loc}}(e_Z)$, let $\mathfrak{S}(w)$ be the set of couples of sequences of parts of $\{1, \ldots, d\}$, $\big((S_0^-, \ldots, S_u^-), (S_0^+, \ldots, S_u^+)\big)$, $u \in \mathbb{N}^*$ such that :
i) $(S_0^-, S_0^+) = (S^-, S^+)$
ii) For each $i \geq 2$, we have $S_{i-1}^+ \neq \emptyset$, and there exists a subset P of ∂S_{i-1}^- such that $(S_i^-, S_{i+1}^+) = (S_i^- - P, P)$.

There is a natural graphical way to visualize $\mathfrak{S}(w)$ by the following object :

Definition 5.2.9. For $w \in \mathcal{W}^{\mathrm{loc}}(e_Z)$, let $\mathrm{Tree}(w)$ be the tree built inductively as follows :
- the root of the tree is labeled by $(S^-(w), S^+(w))$
- consider a vertex of the tree labeled by a couple of parts (E^-, E^+) of $\{1, \ldots, d\}$. If $E^+ \neq \emptyset$, then, for each part $P \subset \partial S^-(w)$, we draw an arrow starting from V to a new vertex V', and we label V' by the couple $(E^- - P, P)$.

The process defining $\mathrm{Tree}(w)$ stops after a finite number of steps by the argument given above ensuring the well-definedness of el, and we have a natural bijection

$$\mathfrak{S}(w) \xrightarrow{\sim} \{\text{ paths from the root to the leaves of } \mathrm{Tree}(w) \text{ }\}$$

Proposition 5.2.10. (rough statement) For all w, $\mathrm{el}(w)$ is given by an explicit sum indexed by $\mathfrak{S}(w)$.

We stress that the terms of the formulas are explicit by means of the operations of taking subwords, quotient words, boundary ∂, connected components, and operations on rational functions : the projections on the positive and negative parts of rational functions of §5.2.2, the composition of rational functions, and the following operations :

Definition 5.2.11. Let the operator of demutiplication of variables $\mathbb{Q}(\xi)[T_{\inf}, T_{\sup}] \to \mathbb{Q}(\xi)[(T_{\inf,i})_{i\in\mathbb{N}}, (T_{\sup,j})_{j\in\mathbb{N}}]$ defined as the linear map that sends $T^i_{\inf}T^j_{\sup} \mapsto T_{\inf,i}T_{\sup,j}$ for all i, j

5.3. **Multiplication of the upper bound by an integer.** Let $\mu, n \in \mathbb{N}^*$. We want to know whether multiple harmonic sums of upper bound $n\mu$ are related in some way to multiple harmonic sums of upper bounds n and μ. We will write the solution in two steps, where only the second step is p-adic.

5.3.1. *Formula in* $\mathbb{Q}(\xi)$. Taking a multiple harmonic sum indexed by $0 < n_1 < \ldots < n_d < \mu n$, we partition the set of all possible $(n_1, \ldots, n_d)$ in function of their divisibility or non-divisibility by μ, and we regroup the indices that are not divisible by μ in function of the quotient in their Euclidean division by μ : this gives a formula involving multiple harmonic sums such that the difference between the upper and lower bounds is equal to μ.

Definition 5.3.1. Let $d \in \mathbb{N}^*$. Let $\mathcal{I}_d$ be the set of triples

$$(E_{|}, E_{\nmid}, F(E_{\nmid}))$$

where $E_{|}, E_{\nmid}$ are subsets of $\{1, \ldots, d\}$ forming a partition of $\{1, \ldots, d\}$, and F is a connected partition of each element of $C_{E\nmid}$.

Definition 5.3.2. Let $\mu, d \in \mathbb{N}^*$. Let

$$eucl_{\mu,d} : \{(n_1, \ldots, n_d) \in (\mathbb{N}^*)^d \mid 0 < n_1 < \ldots < n_d < \mu n\} \qquad \longrightarrow \mathcal{I}_d$$

be the map which associates with a $(n_1, \ldots, n_d)$ the element $(E_{|}, E_{\nmid}, F(E_{\nmid})) \in \mathcal{I}_d$ such that $E_{|}$, resp. $E_{\nmid}$ is the set of the elements $i \in \{1, \ldots, d\}$ such that n_i is divisible, resp. not divisible, by μ, and $F(E_{\nmid})$ is the connected partition of $E_{\nmid}$ determined by the quotient of the Euclidean division of the integers n_i, $i \in E_{\nmid}$, by μ.

Proposition 5.3.3. $eucl_{\mu,d}$ is surjective.

Proof. Clear. □

In particular, the data of $\mathcal{I}_d$ amounts to a partition of the sets of the form $\{(n_1, \ldots, n_d) \in (\mathbb{N}^*)^d \mid 0 < n_1 < \ldots < n_d < \mu n\}$, $\mu \in \mathbb{N}^*$, defined by the value of $eucl_{\mu,d}$.

Example 5.3.4. i) $\mathcal{I}_1$ has two elements, corresponding to the partition of $\{n_1 \in \mathbb{N}^* \mid 0 < n_1 < \mu n\}$ into the subsets characterized, respectively, by $\mu | n_1$ and $\mu \nmid n_1$
ii) $\mathcal{I}_2$ has five elements, corresponding to partition of $\{(n_1, n_2) \in (\mathbb{N}^*)^2 \mid 0 < n_1 < n_2 < n\}$ into the subsets characterized by : ($\mu | n_1$ and $\mu | n_2$), ($\mu \nmid n_1$,$\mu \nmid n_2$ and $[\frac{n_1}{\mu}] = [\frac{n_2}{\mu}]$), ($\mu \nmid n_1$,$\mu \nmid n_2$ and $[\frac{n_1}{\mu}] < [\frac{n_2}{\mu}]$), ($\mu | n_1$ and $\mu \nmid n_2$) ($\mu \nmid n_1$ and $\mu | n_2$)

Notations 5.3.5. i) If $I \subset \{1, \ldots, d\}$, for each index $\left(\begin{array}{c} y_{d+1}, \ldots, y_1 \\ \chi_d, \ldots, \chi_1 \end{array}\right)$, we denote by $\left(\begin{array}{c} y_d, \ldots, y_1 \\ \chi_d, \ldots, \chi_1 \end{array}\right)|_I$ its subsequence consisting of the indices y_i and χ_i with $i \in I$.
ii) if $U = [a, b] \subset \{1, \ldots, d\}$, and if $n_{i_1} < \ldots < n_{i_d}$ is an increasing sequence of d elements of $\mathbb{N}$, let $\mathrm{har}_{n_U} = \mathrm{har}_{a,b}$.

We can now write a first expression of the effect of the multiplication by μ of the upper bound of a multiple harmonic sum.

Proposition 5.3.6. For all words in $\mathcal{W}_{*,d}(e_Z)$, we have :

$$\text{har}_{\mu n}\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right) = \sum_{\substack{(E_{|},E_{\dagger},F(E_{\dagger}))\in\mathcal{I}_d \\ (n_1,\ldots,n_d)\in eucl^{-1}_{\mu,d}((E_{|},E_{\dagger},F(E_{\dagger})))}} \prod_{V\in C_{E\dagger}} \mu^{\text{weight}(\chi_i)}\chi_i(\mu) \tag{5.3.1}$$

$$\prod_{V\in C_{E\dagger}} \text{har}_{n_V}\left(\begin{array}{c} z^{\mu}_{i_{d+1}},\ldots,z^{\mu}_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)|_V \prod_{U\in F(C_{E_{\dagger}})} \text{har}_{n_U}\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)|_U$$

Proof. By the previous definitions, ths amouns to regrouping the indices $(n_1,\ldots,n_d)$ in function of the values of the quotients of their euclidean division by P and the fact that they divide P or not. □

5.3.2. *Formula in K in the prime case.* In this part we assume that $\mu = p^{\alpha}$. We are going to remove the dependence in $(n_1,\ldots,n_d) \in \phi^{-1}((E_{|},E_{\dagger},F(E_{\dagger})))$ in the previous sum via a p-adic expansion.

Proposition-Definition 5.3.7. There exists an explicit map $(\circ^{\text{RT,RT}}_{\text{har}})_{loc} : K\langle\langle e_Z\rangle\rangle_{\Sigma}\times\text{Map}(\mathbb{N},K\langle\langle e^{\text{loc}}_Z\rangle\rangle) \to \text{Map}(\mathbb{N},K\langle\langle e_Z\rangle\rangle)$ such that for all $w \in W^{loc}(e_Z)$, we have

$$(\text{har}_{p^{\alpha}n})_{w\in W^{loc}/X_K} = (har_{p^{\alpha}})_{w/X_K}(\circ^{\text{RT,RT}}_{\text{har}})_{loc}(\text{har}_n)_{w\in W^{loc}/X_K^{(p^{\alpha})}}$$

Proof. In the previous Proposition, we consider the indices n_i that are non-divisible by p^{α}, and we apply the Lemma of shifting of multiple harmonic sums (§4.2.3). This amounts to write the Euclidean division of n_i by p^{α}, as $n_i = p^{\alpha}u_i + r_i$, to write $\frac{1}{n_i^{s_i}} = \frac{1}{r_i^{s_i}}\sum_{l_i\geq 0}\binom{-s_i}{l_i}\left(\frac{p^{\alpha}u_i}{r_i}\right)^{l_i}$. Then, we separate the summations over u_i's and over r_i's ; the summation over u_i's give multiple harmonic sums $\text{har}_n(w)$ with $w \in \mathcal{W}^{loc}(e_Z)$; the summation over r_i's gives prime multiple harmonic sums $\text{har}_{p^{\alpha}}$. □

Once again, this map can be written entirely explicitly. We give only the first examples ; further examples require the introduction of certain combinatorial tools.

Example 5.3.8. In depth one and two and for $\mathbb{P}^1 - \{0,1,\infty\}$, we obtain :

$$\text{har}_{p^{\alpha}n}(s) = \text{har}_n(s) + \sum_{l_1\in\mathbb{N}} n^s\tilde{\sigma}_{l_1}(n)\binom{-s}{l_1}\text{har}_{p^{\alpha}}(s+l_1)$$

$$\begin{aligned}\text{har}_{p^{\alpha}n}(s_2,s_1) &= \text{har}_n(s_2,s_1) \\ &+ \sum_{l_1,l_2\geq 0}\prod_{i=1}^{2}\binom{-s_i}{l_i}n^{s_i}\times\Big[\tilde{\sigma}_n(-l_1-l_2)\,\text{har}_{p^{\alpha}}(s_2+l_2,s_1+l_1) \\ &\qquad + \tilde{\sigma}_n(-l_2,-l_1)\prod_{i=1}^{2}\text{har}_{p^{\alpha}}(s_i+l_i)\Big] \\ &+ n^{s_1+s_2}\Big[\sum_{l_1\geq 0}\text{har}_{p^{\alpha}}(s_1+l_1)\binom{-s_1}{l_1}\tilde{\sigma}_n(s_2,-l_1) - \sum_{l_2\geq 0}\text{har}_{p^{\alpha}}(s_2+l_2)\binom{-s_2}{l_2}\tilde{\sigma}_n(s_1,-l_2)\Big] \\ &+ \text{har}_n(s_2)\sum_{l_1\geq 0}\tilde{\sigma}_n(l_1)\binom{-s_1}{l_1}\text{har}_{p^{\alpha}}(s_1+l_1)\end{aligned} \tag{5.3.2}$$

5.4. **The rational harmonic Ihara action.**

Proposition-Definition 5.4.1. Let the rational harmonic Ihara action $\circ_{\text{har}}^{\text{RT,RT}} : K\langle\langle e_Z\rangle\rangle_\Sigma \times \text{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle) \to \text{Map}(\mathbb{N}, K\langle\langle e_Z\rangle\rangle)$ as the post-composition of the localized rational harmonic Ihara action $(\circ_{har}^{\text{RT,RT}})_{\text{loc}}$ (of §5.3) by the dual of the elimination of postive powers (id $\times el$) (of §5.2).
Then, we have

$$(\text{har}_{p^\alpha\mathbb{N}} = (\text{har}_{p^\alpha})_{w/X_K} \circ_{\text{har}}^{\text{RT,RT}} (\text{har}_{\mathbb{N}}^{(p^\alpha)}$$

Let us write the first examples of the formula for $\circ_{\text{har}}^{\text{RT,RT}}$.

Example 5.4.2. In depth one and two and for $\mathbb{P}^1 - \{0, 1, \infty\}$, we have :
i) (depth one) For all $s \in \mathbb{N}^*$, we have :

$$\text{har}_{p^\alpha n}(s) = \text{har}_n(s) + \sum_{l\geq 1} n^{s+l} \sum_{l_1 \geq l-1} \mathcal{B}_l^{l_1} \binom{-s}{l_1} \text{har}_{p^\alpha}(s + l_1)$$

In particular, the rational harmonic Ihara action in depth one is given by

$$(g \circ_{\text{har}}^{\text{RT}} h)_n(s) = h_n(s) + \sum_{l\geq 1} n^{s+l} \sum_{l_1\geq l-1} \mathcal{B}_l^{l_1} \binom{-s}{l_1} g(s + l_1)$$

ii) (depth two) For all $s_2, s_1 \in \mathbb{N}^*$, we have :

$$\text{(5.4.1)} \quad \text{har}_{p^\alpha n}(s_2, s_1) = \text{har}_n(s_2, s_1)+$$

$$\sum_{t\geq 1} n^{s_1+s_2+t} \Big[\sum_{\substack{l_1,l_2\geq 0\\ l_1+l_2\geq t-1}} \mathcal{B}_t^{l_1+l_2} \prod_{i=1}^{2} \binom{-s_i}{l_i} \text{har}_{p^\alpha}(s_2+l_2, s_1+l_1) + \sum_{\substack{l_1,l_2\geq 0\\ l_1+l_2\geq t-2}} \mathcal{B}_t^{l_2,l_1} \prod_{i=1}^{2} \binom{-s_i}{l_i} \text{har}_{p^\alpha}(s_i+l_i) \Big]$$

$$+ \sum_{\substack{1\leq t\leq s_2-1\\ l_1\geq t-1}} n^{s_1+t} \text{har}_n(s_2-t) \mathcal{B}_t^{l_1} \binom{-s_1}{l_1} \text{har}_{p^\alpha}(s_1+l_1) - \sum_{\substack{1\leq t\leq s_1-1\\ l_2\geq t-1}} n^{s_2+t} \text{har}_n(s_1-t) \mathcal{B}_t^{l_2} \binom{-s_2}{l_2} \text{har}_{p^\alpha}(s_2+l_2)$$

$$- n^{s_2+s_1} \Bigg[\sum_{l_1\geq s_2-1} \mathcal{B}_{s_2}^{l_1} \binom{-s_1}{l_1} \text{har}_{p^\alpha}(s_1 + l_1) - \sum_{l_2\geq s_1-1} \mathcal{B}_{s_1}^{l_2} \binom{-s_2}{l_2} \text{har}_{p^\alpha}(s_2 + l_2) \Bigg]$$

$$+ \sum_{t'\geq 1} n^{t'} \Bigg[\sum_{\substack{t\geq s_2+t'-1\\ l_1\geq t-1}} \mathcal{B}_{t'}^{t-s_2} \mathcal{B}_t^{l_1} \binom{-s_1}{l_1} \text{har}_{p^\alpha}(s_1 + l_1) - \sum_{\substack{t\geq s_1+t'-1\\ l_2\geq t-1}} \mathcal{B}_{t'}^{t-s_1} \mathcal{B}_t^{l_2} \binom{-s_2}{l_2} \text{har}_{p^\alpha}(s_2 + l_2) \Bigg]$$

5.5. **A lift and a generalization of the rational harmonic Ihara action.** We can lift the rational harmonic Ihara action into a formula that reflects the Ihara action on group of analytic sections of U^{an} of the bundle of paths starting at $\vec{1}_0$ by the two following facts.

First, the formula for $\text{har}_{p^\alpha n}$ in terms of har_{p^α} and har_n expressed by $\circ_{\text{har}}^{\text{RT,RT}}$ can be lifted to a formula of the following type, for all words w :

$$\text{(5.5.1)} \quad \text{har}_{p^\alpha n}(w) = \sum_{m=0}^{n} \Big(\text{ function of } \text{har}_m \text{ and } \text{har}_{p^\alpha} \Big) . \Big(\text{ analytic function of (m,n) } \Big)$$

This fact can be proved by adapting the computations of §5.2 and §5.3. Actually, for each word w, there are several possible ways to define m such that an expression of $\text{har}_{p^\alpha n}(w)$ as above holds. We want moreover that the right-hand side factorizes as an analytic function of $n - m$. Finally in the right-hand side, the coefficients of these analytic functions are infinite sums of

har_{p^α}.
We will discuss the possible ways to choose m as above in the final version of this paper. In the meantime, we will only explain in §6.3 the relation between the formulas such as (5.5.1) and the De Rham setting.

Secondly, one can generalize the formula for $\mathrm{har}_{p^\alpha n}$ given by $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$, into an expression of all multiple harmonic sums $\mathrm{har}_{r+p^\alpha n}$ for all $r \in \{0,\dots,p^\alpha - 1\}$. Indeed, we have :

Lemma 5.5.1. Let $r \in \{1,\dots,p^\alpha-1\}$. Then, for all w, $\mathrm{har}_{r+p^\alpha n}[w]$ is a polynomial of values of $\mathrm{har}_{p^\alpha n}$ and of analytic functions of $p^\alpha n$ whose coefficients are expressed in terms of har_r.

Proof. We apply the formula of splitting at $p^\alpha n$ (§4.2.2) to express $\mathrm{har}_{p^\alpha n+r}$ in terms of $\mathrm{har}_{p^\alpha n}$ and $\widetilde{\mathrm{har}}_{p^\alpha n, p^\alpha n+r}$; then, the formula of shifting (§4.2.3) to express $\mathrm{har}_{p^\alpha n,p^\alpha n+r}$ as an analytic function of $p^\alpha n$ with coefficients expressed in terms of har_r. □

6. Comparison between the De Rham and the rational actions

In §6.1, we define an "absolute" harmonic Ihara action and show that it factorizes both De Rham and rational harmonic Ihara actions. In §6.2, we deduce the De Rham-rational comparison for the harmonic Ihara action and finish the proof of Theorem I-2.a. In §6.3, we deduce the De Rham-rational comparison for the Ihara action on the bundle of paths starting at $\vec{1}_0$ and finish the proof of Theorem I-2.b. In §6.4, we discuss some other De Rham-rational comparisons.

6.1. The absolute harmonic Ihara action $\circ_{\mathrm{har}}^{\mathrm{abs}}$ over $\mathbb{Q}$ and relation with $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ and $\circ_{\mathrm{har}}^{\mathrm{RT}}$.

6.1.1. *Definition.* The absolute harmonic Ihara action can be defined over $\mathbb{Q}$ and is a natural intermediate object to relate $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$ and $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$. In some sense, it plays the role of a motivic Galois action that lies above two realizations.

Definition 6.1.1. i) Let $\mathcal{W}^{abs}(e_Z)$ be, when $N=1$, the following set of indices :

$$\{(s_d,\dots,s_1|s_0) \mid s_d,\dots,s_0 \in \mathbb{N}^*\} \cup \{(\infty|s_d,\dots,s_1|s_0) \mid s_d,\dots,s_0 \in \mathbb{N}^*\}$$

and similarly for any N.
ii) Let $\mathcal{W}^{abs}_{conv}(e_Z) \subset \mathcal{W}^{abs}(e_Z)$ be the subset consisting of the indices as above with $s_0 = 1$; they will be denoted as follows

$$(s_d,\dots,s_1) = (s_d,\dots,s_1|1)$$
$$(\infty|s_d,\dots,s_1) = (\infty|s_d,\dots,s_1|1)$$

Definition 6.1.2. Let the absolute harmonic Ihara action be the map

$$\circ_{\mathrm{har}}^{abs} : K\langle\langle W^{abs}(e_Z)\rangle\rangle \times K\langle\langle W^{abs}_{conv}(e_Z)\rangle\rangle \to K\langle\langle W^{abs}_{conv}(e_Z)\rangle\rangle$$

defined by :

$$(g \circ_{\mathrm{har}}^{abs} h)(s_d,\dots,s_1) = \sum_{0=i_0\leq j_0<i_1<j_1<\dots<i_r\leq j_r=d} g(\infty|s_{i_d},\dots,s_{i_r}|t_{i_r-1})\times$$
$$\prod_{k=0}^{r-1} h(s_{j_k}-t_{j_k}, s_{j_k-1},\dots,s_{i_k+1}s_{i_k}-t_{i_k}) g(t_{i_{k+1}}, s_{j_{k+1}},\dots,s_{i_{k+1}-1}|t_{j_k})$$

Example 6.1.3. For $\mathbb{P}^1 - \{0,1,\infty\}$, in depth one we have :

$$(f \circ_{\mathrm{har}}^{abs} h)(s) = h(s) + f(\infty|s)$$

In depth two, we have :

$$(f \circ_{\mathrm{har}}^{abs} h)(s_2,s_1) = h(s_2,s_1) + f(\infty|s_2,s_1) + \sum_{r_2=0}^{s_2-1} h(s_2-r_2) f(r_2+1,s_1) - \sum_{r_1=0}^{s_1} h(s_1-r_1) f(\infty|s_2|r_1)$$

6.1.2. *Relation with the De Rham-rational harmonic Ihara action.* We now see that the absolute harmonic Ihara action provides a way to reindex $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$.

Lemma 6.1.4. For all $g \in \tilde{\Pi}_{1,0}(K)$, there exists g_{abs} such that for all f, we have

$$g \circ_{\mathrm{har}}^{\mathrm{DR,RT}} f = g_{\mathrm{abs}} \circ_{\mathrm{har}}^{\mathrm{abs}} f$$

Indeed g_{abs} is given by the formula that we can read by comparing the definitions of $\circ_{\mathrm{har}}^{\mathrm{abs}}$ and $\circ_{\mathrm{har}}^{\mathrm{DR,RT}}$.

6.1.3. *Relation with the rational harmonic Ihara action.* We now see that the absolute harmonic Ihara action provides a way to reindex $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$ as well.

Lemma 6.1.5. For all g such that $\Sigma^{\mathrm{RT}}(g)$ defines a element $\tilde{\Pi}_{1,0}(K)$, there exists g_{abs} such that for all f, we have

$$g \circ_{\mathrm{har}}^{\mathrm{RT,RT}} f = g_{\mathrm{abs}} \circ_{\mathrm{har}}^{\mathrm{abs}} f$$

Proof. Namely, $g(m|s_d,\ldots,s_1)$ is the coefficient of $n^{m+s_d+\ldots+s_1}$ in $g \circ_{\mathrm{har}}^{\mathrm{RT,RT}} f$ for all f ; and we have a similar definition for the other coefficients. The result can be proved by looking closely at the definition of $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$; roughly speaking, since the definition of $\circ_{\mathrm{har}}^{\mathrm{RT,RT}}$ is made by means of rational analogues of subwords and quotient words, it is of the same nature with the usual Ihara action. ☐

6.2. **Comparison at $(\vec{1}_1, \vec{1}_0)$.** In order to finish the proof of Theorem I-2.a, we have to prove in particular the injectivity of the map Σ^{RT}. It seems to us that the best way to prove it is the following.

Proposition 6.2.1. We have ${\Sigma^{\mathrm{DR}}}^{-1} \circ \Sigma^{\mathrm{RT}} = \mathrm{id}$. In particular, the map Σ^{RT} is injective.

Proof. Essentially follows from the definition of Σ^{RT}, ${\Sigma^{\mathrm{DR}}}^{-1}$ and the following lemma. ☐

Lemma 6.2.2. i) The coefficients $\mathcal{B}$ of §4.1 satisfy an algebraic relation that amounts to say that multiple harmonic sums of upper bound 1 associated with non-empty words vanish.
ii) This is more generally true for the variants of the coefficients B expressing the polynomial part of the map of elimination of positive powers.

We can thus define :

Definition 6.2.3. Let $(\pi_1^{\mathrm{un,RT}}(K), \circ^{\mathrm{RT}})$ be the inverse image of $\pi_1^{\mathrm{un,DR}}(K)$ by Σ^{RT}.

We leave to the reader to show that these facts imply the rest of Theorem I-2.a and that the last proposition gives also a proof of Corollary I-2.a that differs from the one in §3. The rational harmonic Ihara action is defined without any analogue of the condition for an element of $K\langle\langle e_Z\rangle\rangle$ to be grouplike (this deficiency will be repaired in II). In order to define the objects appearing in Theorem I-2.a, we just have to restrict to elements g such that the image by Σ^{RT} defines a grouplike series of $K\langle\langle e_Z\rangle\rangle$, in which case the rational harmonic Ihara action of g factors through the De Rham-rational harmonic Ihara action of g. We know that these objects are non-empty because they contains the points defined by $\zeta_{p,\alpha}$, har_{p^α}, $\mathrm{har}_{\mathbb{N}}^{(p^\alpha)}$ and $\mathrm{har}_{p^\alpha\mathbb{N}}$.

6.3. **Comparison on the bundle of paths starting at $\vec{1}_0$.** The decomposition of multiple harmonic sums in §5.5 depended on the choice of a parameter m, and their was several possible choices for m. The possible values are parametrized by certain words in the De Rham setting as follows. We consider words $e_0^l e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}}$. For each l, we have a definition of the parameter m of §5.5 associated with the multiple harmonic sums of $\left(\begin{array}{c} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{array} \right)$. This can be seen by considering the equation of horizonality to which we apply $\tau(n)[z^n]$ as in §2. We recall that, by I-1, the map $n \in \mathbb{N}^* \mapsto \mathrm{Li}^\dagger_{p,\alpha}[z^n]$ extends to a locally analytic map $\mathbb{Z}_p \to \mathbb{Q}_p$ that is analytic on each closed disk of radius p^α.

Thus, in order to finish the proof of Theorem I-2.b, we are essentially reduced to showing an analogue of the Proposition 6.2.1.
It follows from the horizontality equation that we have

Lemma 6.3.1. For all $r \in \{1, \dots, p^\alpha - 1\}$, we have

$$\mathrm{Li}^\dagger_{p,\alpha}[z^r][w_l^{(p^\alpha)}] = p^{\mathrm{weight}(w)+l} r^l \,\mathrm{har}_r(w)$$

Similarly, in the rational setting, the decomposition of har_r in the sense of §5.5 is trivial. The combination of those two facts gives the desired information.
Note that we also have

$$\mathrm{Li}^\dagger_{p,\alpha}[w_l][z^{p^\alpha}] = (-1)^{d+1}\,\mathrm{har}_{p^\alpha}(\tilde{w}) - \sum_{l'=0}^{l-1} \sum_{i=1}^{N} -z_i^{-p^\alpha} \left({\Phi^{(z_i)}_{p,\alpha}}^{-1} e_{z_i} \Phi^{(z_i)}_{p,\alpha} \right)[w_{l'}]$$

We observe in particular that the left-hand side, that mixes objects of different arithmetic natures (multiple harmonic sums and K-linear combinations of values of p-adic hyperlogarithms) and might not look natural, has actually a very simple and natural expression.

6.4. **Other comparisons between the De Rham and rational setting.**

6.4.1. *Multiple harmonic sums at non-convergent indices and shifting.* In §3.4, we encountered in the De Rham setting the following linear combinations of prime weighted multiple harmonic sums

$$\sum_{\substack{r_d,\dots,r_1 \geq 0 \\ r_d+\dots+r_1 = r'}} \prod_{i=1}^{d} \binom{-s_i}{r_i} \mathrm{har}_{p^\alpha} \left(\begin{array}{c} z_{i_{d+1}}, \dots, z_1 \\ s_d + r_d, \dots, s_1 + r_1 \end{array} \right)$$

They were expressed as infinite sums of p-adic multiple zeta values at words that were not necessarily convergent at 0. Their interpretation in the rational setting is the following : they are equivalent to a shifted variant of multiple harmonic sums :

Remark 6.4.1. For each $n \in \mathbb{N}^*$, and for each $\lambda \in K$ close enough to zero, we have, for all indices,

$$\sum_{r' \in \mathbb{N}} \lambda^{r'} \sum_{\substack{r_d,\dots,r_1 \geq 0 \\ r_d+\dots+r_1 = r'}} \prod_{i=1}^{d} \binom{-s_i}{r_i} \mathrm{har}_n \left(\begin{array}{c} z_{i_{d+1}}, \dots, z_1 \\ s_d + r_d, \dots, s_1 + r_1 \end{array} \right)$$
$$= n^{s_d+\dots+s_1} \sum_{0<n_1<\dots<n_d<n} \frac{\left(\frac{z_{i_2}}{z_{i_1}}\right)^{n_1} \dots \left(\frac{z_{i_{d+1}}}{z_{i_d}}\right)^{n_d} \left(\frac{1}{z_{i_{d+1}}}\right)^{n}}{(n_1+\lambda n)^{s_1} \dots (n_d + \lambda n)^{s_d}}$$

For non-weighted multiple harmonic sums, the same formula holds with λ instead of λn above. We will study intrinsically these objects in the next parts of the theory.

6.4.2. *Comparison of completed algebras.* Let $\mathcal{P}_N$ be the set of prime numbers that are prime to N. We introduced in I-1 a certain algebra $\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}} \subset \prod_{p \nmid N} \overline{\mathbb{Q}_p}$ of certain infinite sums of linear combinations of prime weighted multiple harmonic sums ; its definition is recalled in §4.3. Having encountered certain infinite sums of p-adic multiple zeta values, it is now natural to introduce the following analogue :

Definition 6.4.2. Let

$$\widehat{Z}_{\mathcal{P}_N,\mathbb{N}^*} = \mathrm{Vect}_{\mathbb{Q}}\Big\{ \Big(\sum_{L \in \mathbb{N}} F_L(\xi,\ldots,\xi^{N-1}) \prod_{\eta=1}^{\eta_0} \zeta_{p,\alpha}(w_{L,\eta}) \Big)_{(p,\alpha)} \mid (*) \Big\} \subset \prod_{\substack{(p,\alpha) \\ \in \mathcal{P}_N \times \mathbb{N}^*}} \mathbb{Q}_p(\mu_N)$$

where $(*)$ means that $(w_{L,1})_{L\in\mathbb{N}},\ldots,(w_{L,\eta_0})_{L\in\mathbb{N}}$ ($\eta_0 \in \mathbb{N}^*$) are sequences of words satisfying $\sum_{\eta=1}^{\eta_0} \mathrm{weight}(w_{L,\eta}) \rightarrow_{l\rightarrow\infty} \infty$ and $\limsup_{L\rightarrow\infty} \sum_{\eta=1}^{\eta_0} \mathrm{depth}(w_{L,\eta}) < \infty$, and $(F_L)_{L\in\mathbb{N}}$ is a sequence of elements of $\mathbb{Q}$ if $N=1$, resp. $\mathbb{Q}[T_1,\ldots,T_{N-1},\frac{1}{T_1},\ldots,\frac{1}{T_{N-1}},\frac{1}{T_1-1},\ldots,\frac{1}{T_{N-1}-1}]$ if $N \neq 1$.

In I-1 we proved that $\zeta_{p,\alpha}(w)$ are elements of $\widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}$. This implies in particular the inclusion $\widehat{Z}_{\mathcal{P}_N,\mathbb{N}^*} \subset \widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}$. By the Corollary I-2.a, we have now the converse inclusion. Whence :

Corollary 6.4.3. We have $\widehat{Z}_{\mathcal{P}_N,\mathbb{N}^*} = \widehat{\mathrm{Har}}_{\mathcal{P}_N^{\mathbb{N}^*}}$.

We note that the second proof of the Corollary I-2.a, that goes through the rational setting, shows that the expansion of har_{p^α} in terms of $\zeta_{p,\alpha}$ can be obtained from inverting the expansion of $\zeta_{p,\alpha}$ in terms of har_{p^α}.

6.4.3. *Shape of the rational coefficients.* The computations of I-2 enlighten some other combinatorial aspects of cyclotomic p-adic multiple zeta values. As we have seen in I-1 and I-2, the sums of series expressing cyclotomic p-adic multiple zeta values in terms of prime weighted multiple harmonic sums involve rational coefficients (independent of p and α), made out of two types of coefficients : the coefficients of the polynomials expressing sums of powers of natural integers multiplied by exponential functions, $\mathcal{B}^l_m(z_i)$ (for $z=1$, this is $\frac{1}{l+1}\binom{l+1}{m}B_{l+1-m}$ where B denotes Bernoulli numbers) and the generalized binomial coefficients $\binom{-s}{l} = (-1)^l\binom{l+s-1}{s-1}$, $s \in \mathbb{N}^*$, $l \in \mathbb{N}$.
Those two types of coefficients appeared at the same time and were mixed together in the formulas of I-1. Here, in I-2, they appear separated from each other : the coefficients $\mathcal{B}$ appear in the elimination of positive powers of multiple harmonic sums (§5.2) and the binomial coefficients appear in the p-adic expansions of shifted multiple harmonic sums (§5.3). The rational harmonic Ihara action is the composition of two maps, one involving uniquely the coefficients $\mathcal{B}$ and the other one uniquely the binomial coefficients.
Moreover, we see in this I-2, by §5.1 and §5.2, that it is convenient to write the formulas, not in terms of the numbers $\mathcal{B}^l_m$, but in terms of certain polynomials of these numbers.

6.4.4. *Announcements.* The final version of this paper will include the explicit formulas that are only implicit in §5 and §6 of this version. The final version of this paper will also include the following facts. The main results of this paper remain true in the ramified case, i.e. when $p|N$. Finally, some of them also have a variant where $z_{i_1},\ldots,z_{i_n}$ are replaced by formal variables and by more general p-adic variables in $\{z \in \mathbb{C}_p \mid |z|_p = 1\}$.

We will explain also later the generalization of these facts to the variant of multiple harmonic

sums defined as follows : for each $d \in \mathbb{N}^*$, choose a subset J of $\{1, \ldots, d\}$, then a strictly increasing map $a : J \mapsto \mathbb{N}^*$, and restrict the iterated summation $0 < n_1 < \ldots < n_d < n$ defining multiple harmonic sums to indices such that $n_j = a(j)$ for all $j \in J$. We will define the harmonic Frobenius for these generalized multiple harmonic sums and explain its De Rham interpretation.

In II, we will investigate the consequences of our computations regarding algebraic relations. Roughly speaking, we will prove that the harmonic Ihara action is compatible with algebraic relations, especially the double shuffle relations, and that it induces a correspondence between the algebraic relations of $\zeta_{p,\alpha}$ and some algebraic relations satisfied by weighted multiple harmonic sums. This will give a way to visualize explicitly the algebraic relations of p-adic multiple zeta values, since multiple harmonic sums are explicit. Below we discuss only double shuffle relations, but in II-1 and II-2 we will also discuss some consequences of Drinfeld associator and Kashiwara-Vergne relations.

The prime weighted multiple harmonic sums har_{p^α} deserve a study of their own and we will treat them first, in II-1. There, we will define and study a variant of double shuffle relations, that we will call prime harmonic double shuffle relations, that they will be shown to satisfy. We will study har_{p^α} in three different ways. The first way is in the De Rham setting ; we will use the expression of har_{p^α} provided by Corollary I-2.a and show that the map $\Sigma^{DR^{-1}}$ sends solutions to the usual double shuffle relations to solutions of the prime harmonic double shuffle relations. The second way will be in what we call the De Rham-rational setting, that is to say we will use the expression of har_{p^α} in terms of the series expansion of hyperlogarithms. The third way will be in what we call the rational setting, that is to say we will use using exclusively multiple harmonic sums, and some information on their Newton series. This third type of computation will be mostly an interpretation of some work by Hoffman and Rosen. We will see that the three settings will give the same results.

In II-2, we will have to use the double shuffle relations satisfied by all multiple harmonic sums har_n : we note that the integral shuffle relation involves the generalization of har_n defined as

$$\sum_{0<n_1<\ldots<n_d<n'<m_{d'}<\ldots<m_1<n} \frac{\left(\frac{z_{i_2}}{z_{i_1}}\right)^{n_1} \ldots \left(\frac{z_{i_{d+1}}}{z_{i_d}}\right)^{n_d} \left(\frac{z_{j_{d'+1}}}{z_{i_{d+1}}}\right)^{n'} \left(\frac{z_{j_{d'}}}{z_{j_{d+1}}}\right)^{m_{d'}} \ldots \left(\frac{z_{j_2}}{z_{j_1}}\right)^{m_1}}{n_1^{s_1} \ldots n_d^{s_d} (n-m_{d'})^{t_{d'}} \ldots (n-m_1)^{t_1}}$$

We will see first that the rational part Theorem I-2.a has a natural extension to this generalization of multiple harmonic sums. Our first task will then be to find a nice De Rham expression of the p-adic coefficients that will appear, that would be coherent with their integral shuffle relation. Let us call harmonic double shuffle relations the variant of double shuffle relations satisfied by $\mathrm{har}_{\mathbb{N}}$. We will prove that, if f is a solution to the harmonic double shuffle relations satisfying certain conditions, then we have an equivalence as follows : $g \circ_{\mathrm{har}} f$ satisfies the harmonic double shuffle relations if and only if g satisfies the usual double shuffle relations. This is a variant of Racinet's theorem on the double shuffle relation and the Ihara action.
We will use a key lemma a property of linear independence of the functions $n \in \mathbb{N} \mapsto \mathrm{har}_n[w] \in \mathbb{Q}_p$ over a certain ring of functions of n. This property will turn out to be stable by the harmonic Ihara action, and for f satisfying this property, if $g \circ_{\mathrm{har}} f$ and f satisfy the harmonic double shuffle relations, we can identify the coefficients of each $f[w]$ in the formula for $g \circ_{\mathrm{har}} f$, leading to algebraic relations satisfied by the coefficients of g.
A second result in II-2 will be that the map Σ^{RT} sends solutions to the prime harmonic double

shuffle relations to the solutions to usual double shuffle relations.
The results will use certain algebraic properties of the coefficients $\mathcal{B}$ that we have introduced, expressing algebraic relations between multiple harmonic sums depending on positive and negative powers of the indices ; among them are : the series shuffle relation, a relation obtained by relating in two different ways σ_n and σ_{n-1}, and a relation coming from the reversal of certain indices.
Thus we will have a total of two ways to understand explicitly the algebraic relations of p-adic multiple zeta values, all entirely governed by the (generalized) harmonic Ihara action and its byproducts.
We will also define a "De Rham harmonic Ihara action" $\circ_{\text{har}}^{\text{DR}}$, that is more convenient for certain purposes, and that will be obtained by pushing forward the usual Ihara product along a certain map.
It will be useful for certain purposes to restrict n to elements p^{β} with $\beta \geq \alpha$; by using the prime decomposition of n and considering all prime numbers at the same time, we will also relate the harmonic and prime harmonic double shuffle relations.
In II-3, we will essentialize the computations and objects of I and II and explain more explicitly the concept of rational realization of the pro-unipotent fundamental group in this particular case.

Appendix A. Review of an application to finite MZVs and remarks

A.1. **Finite multiple zeta values.** The notion of finite multiple zeta values has been defined recently by Kaneko and Zagier [KZ]. It makes precise the analogy, observed before by several people, in particular Hoffman and Zhao, between the algebraic properties of multiple harmonic sums H_p modulo p when p is large, and of multiple zeta values. Let $\mathcal{P}$ be the set of prime numbers, and let $\mathcal{A}$ be the ring of integers modulo infinitely large primes, whose first appearance seems to be in Kontsevich's paper [Ko] :

$$\mathcal{A} = \Big(\prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}\Big)/\Big(\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}\Big)$$

We have a natural embedding $\mathbb{Q} \hookrightarrow \mathcal{A}$; more precisely and more categorically, we have :

$$\mathcal{A} = \Big(\prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}\Big)/\Big(\prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}\Big)_{\text{tors}} \simeq \Big(\prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}\Big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where tors refers to the torsion subgroup ; and the ring $\mathcal{A}$ has also an adelic expression :

$$\mathcal{A} = \Big\{(x_p)_p \in \prod_{p\in\mathcal{P}} \mathbb{Q}_p \mid v_p(x_p) \geq 0 \text{ for p large}\Big\} \Big/ \Big\{(x_p)_p \in \prod_{p\in\mathcal{P}} \mathbb{Q}_p \mid v_p(x_p) \geq 1 \text{ for p large}\Big\}$$

Definition A.1.1. (Zagier) Finite multiple zeta values are the following numbers, for $(s_d,\ldots,s_1) \in \amalg_{d'\geq 1}(\mathbb{N}^*)^{d'}$:
(A.1.1)

$$\zeta_{\mathcal{A}}(s_d,\ldots,s_1) = \Big(p^{-(s_d+\ldots+s_1)}\,\text{har}_p(s_d,\ldots,s_1)\Big)_{p\in\mathcal{P}} = \left(\sum_{0<n_1<\ldots<n_d<p} \frac{1}{n_1^{s_1}\ldots n_d^{s_d}}\right)_{p\in\mathcal{P}} \in \mathcal{A}$$

This definition is explained by the following conjecture :

Conjecture A.1.2. i) (Zagier) For $s \in \mathbb{N}$, let $\mathcal{Z}_s$ the $\mathbb{Q}$-vector subspace of $\mathcal{A}$ generated by the finite multiple zeta values of weight s. Then, the sum of the $\mathcal{Z}_s$'s is direct. Moreover, we have $\sum_{s\geq 0} \dim(\mathcal{Z}_s)\Lambda^s = \frac{1-\Lambda^2}{1-\Lambda^2-\Lambda^3}$

ii) (Kaneko-Zagier) The following correspondence defines an isomorphism of $\mathbb{Q}$-algebras from the algebra generated by finite multiple zeta values to the algebra generated by multiple zeta values modulo $(\zeta(2))$ (the right hand side is independent of the chosen regularization) :

$$\zeta_{\mathcal{A}}(s_d, \ldots, s_1) \mapsto \sum_{d'=0}^{d} (-1)^{s_{d'+1}+\ldots+s_d} \zeta(s_{d'+1}, \ldots, s_d)\zeta(s_{d'}, \ldots, s_1) \mod \zeta(2) \tag{A.1.2}$$

An older conjecture is that i) above holds for p-adic multiple zeta values, and also for the image of multiple zeta values modulo $(\zeta(2))$. The Conjecture A.1.2 is striking, both by its analogy with these older conjectures and by the concise explicit formula ii). It has suggested the existence of a relation between finite multiple zeta values and the reduction modulo primes of p-adic multiple zeta values that would be coherent with the formula of ii) ; we now review how this has been obtained, using our Corollary I-2.a.

A.2. **Work of Akagi-Hirose-Yasuda and Chatizmatatiou.** The following theorem follows from the logarithmic generalization of Mazur's theorem of comparison between the Hodge filtration and the Frobenius on crystalline cohomology.

Theorem A.2.1. (Akagi-Hirose-Yasuda (unpublished), Chatizmatatiou [C]) For all indices $(s_d, \ldots, s_1) \in \amalg_{d' \geq 1} (\mathbb{N}^*)^{d'}$, and all primes p, we have :

$$\zeta_p(s_d, \ldots, s_1) \in \sum_{s \geq s_d + \ldots + s_1} \frac{p^s}{s!} \mathbb{Z}_p$$

In particular, when $p > s_1 + \ldots + s_d$, we have

$$\zeta_p(s_d, \ldots, s_1) \in p^{s_d + \ldots + s_1} \mathbb{Z}_p$$

Combining the Corollary I-2.a in the case of $\mathbb{P}^1 - \{0, 1, \infty\}$, $\alpha = 1$, and the Theorem A.2.1 one obtains immediately :

Corollary A.2.2. (Akagi-Hirose-Yasuda) If $p > s_d + \ldots + s_1$, we have :

$$\sum_{0<n_1<\ldots<n_d<p} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}} \equiv$$
$$p^{-(s_d+\ldots+s_1)} \sum_{d'=0}^{d} (-1)^{s_{d'+1}+\ldots+s_d} \zeta_p(s_{d'+1}, \ldots, s_d)\zeta_p(s_{d'}, \ldots, s_1) \mod p$$

In particular, Kaneko-Zagier's finite multiple zeta values can be expressed entirely in terms of p-adic multiple zeta values :

$$\zeta_{\mathcal{A}}(s_d, \ldots, s_1) =$$
$$\Big(p^{-(s_d+\ldots+s_1)} \sum_{d'=0}^{d} (-1)^{s_{d'+1}+\ldots+s_d} \zeta_p(s_{d'+1}, \ldots, s_d)\zeta_p(s_{d'}, \ldots, s_1) \mod p\Big)_{p > s_d + \ldots + s_1} \in \mathcal{A}$$

This explains the explicit formula of equation (A.1.2) in Kaneko-Zagier's conjecture.

Another consequence of the Corollary A.2.2 is that we have a surjective map red_ζ, given by reduction modulo large primes, from the $\mathbb{Q}$-algebra of p-adic multiple zeta values in $\prod_p \mathbb{Z}_p$ onto the $\mathbb{Q}$-algebra of finite multiple zeta values. Then, in fine, Kaneko-Zagier's conjecture reduces to the usual period conjecture on complex and p-adic multiple zeta values combined to :

Conjecture A.2.3. (Kaneko-Zagier's conjecture rephrased) The map red_ζ is an isomorphism.

The existence of the map red_ζ also implies that the dimensions $\dim(\mathcal{Z}_s^{\mathcal{A}})$ of vector spaces of finite multiple zeta values of a given weight are inferior or equal to the analogous dimensions attached to p-adic multiple zeta values in $\prod_p \mathbb{Z}_p$. Since Yamashita has constructed a surjective map from the algebra of motivic multiple zeta values modulo the motivic $\zeta(2)$ onto the algebra of p-adic multiple zeta values (unpublished), this yields :

Corollary A.2.4. (Akagi-Hirose-Yasuda) We have :

$$\sum_{s\geq 0}\dim(\mathcal{Z}_s^{\mathcal{A}})\Lambda^s \leq \frac{1-\Lambda^2}{1-\Lambda^2-\Lambda^3}$$

where the inequality must be understood as the collection of the corresponding inequalities between the coefficients of each Λ^s, $s\in\mathbb{N}$.

A.3. **Remarks.** We would like to suggest a terminology for the images of finite multiple zeta values by Kaneko-Zagier's conjectural isomorphism.

Definition A.3.1. Let us call adjoint multiple zeta values the numbers

$$(\Phi_{\mathrm{KZ}}^{-1}e_1\Phi_{\mathrm{KZ}})[e_1e_0^{s_d-1}e_1\ldots e_0^{s_1-1}e_1] = \sum_{d'=0}^{d}(-1)^{s_{d'+1}+\ldots+s_d}\zeta(s_{d'+1},\ldots,s_d)\zeta(s_{d'},\ldots,s_1)$$

and similarly for their p-adic and motivic analogues, where, $\Phi_{\mathrm{KZ}}\in\pi_1^{\mathrm{un,DR}}(\mathbb{P}^1-\{0,1,\infty\},-\vec{1}_1,\vec{1}_0)(\mathbb{R})$ is the generating series of complex multiple zeta values.

Indeed, the interplay between the algebraic relations satisfied by the coefficients of Φ_{KZ} and $\mathrm{Ad}_{\Phi_{\mathrm{KZ}}}(e_1)=\Phi_{\mathrm{KZ}}^{-1}e_1\Phi_{\mathrm{KZ}}$ is an intrinsically interesting problematic. The exponential of $2i\pi\,\mathrm{Ad}_{\Phi_{\mathrm{KZ}}}(e_1)$ expresses the monodromy of ∇_{KZ}, and enables to define an automorphism μ_{KZ} of the fundamental groupoid of $\mathbb{P}^1-\{0,1,\infty\}$ restricted to certain tangential base-points, such that one has a natural correspondence between the fact that Φ_{KZ} is a Drinfeld associator and the fact that μ_{KZ} is a solution the Kashiwara-Vergne problem [AET]. In II-1 we will prove other properties of adjoint multiple zeta values such as double shuffle relations. Actually, in II we will call adjoint multiple zeta values and study the larger class of numbers

$$(\Phi_{\mathrm{KZ}}^{-1}e_1\Phi_{\mathrm{KZ}})[e_0^{m}e_1e_0^{s_d-1}e_1\ldots e_0^{s_1-1}e_1]$$

as well as their generalization from $N=1$ to any N. We will see that the properties of adjoint multiple zeta values are obtained from those of multiple zeta values by a kind of commutation property between the map $\mathrm{Ad}(e_1)$ and certain algebraic operations ; that finishes to justify our terminology. Of course, these algebraic considerations in the complex setting have motivic and p-adic counterparts.

As a consequence of Corollary A.2.2, we can say that finite multiple zeta values have two expressions relating them to the pro-unipotent fundamental groupoid of $\mathbb{P}^1-\{0,1,\infty\}$: the one given by their definition, that already relates them to coefficients of the series expansion of hyperlogarithms $\mathrm{Li}_p^{\mathrm{KZ}}$, and the one provided by the last corollary, as the reduction of adjoint p-adic multiple zeta values modulo large primes. By our Corollary I-2.a, the existence of two such kinds of expressions actually holds for the following sequences of prime multiple harmonic sums that lift Kaneko-Zagier's finite multiple zeta values :

$$\big(\mathrm{har}_{p^\alpha}(\tilde{w})\big)_{p\in\mathcal{P}}\in\prod_{p\in\mathcal{P}}\mathbb{Z}_p \tag{A.3.1}$$

and this is again true for their generalizations from $N=1$ to any N, that involve a product on the set $\mathcal{P}_N$ of prime numbers that do not divide N. This suggests also that the problematic of studying the algebraic relations between finite multiple zeta values is only a part of the larger one of the algebraic relations between the sequences (A.3.1). We will proceed to such a study in II-1 ; that will allow us to view sequences such as (A.3.1) as periods of the pro-unipotent fundamental groupoid, and that will be one of the building blocks of our explicit algebraic theory of p-adic multiple zeta values. Another perspective, that we will study in IV, is the confrontation of the Theorem A.2.1 and the series expansion of p-adic multiple zeta values.

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Institut de Recherche Mathématique Avancée, Université de Strasbourg, 7 rue René Descartes, 67000 Strasbourg, France

E-mail address: jarossay@math.unistra.fr